THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of THE AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 25, 1899. Forty-seven persons were in attendance, including the following forty members of the Society:

Mr. Joseph Allen, Professor Maxime Bôcher, Professor E. W. Brown, Dr. A. S. Chessin, Dr. J. B. Chittenden, Professor F. N. Cole, Professor T. W. Edmondson, Professor H. B. Fine, Professor T. S. Fiske, Professor James Harkness, Dr. G. W. Hill, Mr. H. E. Hawkes, Professor Harold Jacoby, Mr. C. J. Keyser, Professor Pomeroy Ladue, Professor Gustave Legras, Dr. G. H. Ling, Professor E. O. Lovett, Dr. Emory McClintock, Professor James McMahon, Mr. James Maclay, Professor Alexander Macfarlane, Professor Mansfield Merriman, Dr. G. A. Miller, Professor Frank Morley, Professor W. F. Osgood, Professor A. W. Phillips, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Dr. W. M. Strong, Professor Henry Taber, Professor J. H. Tanner, Professor H. D. Thompson, Professor C. L. Thornburg, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair. The meeting extended, as usual, through a morning and an afternoon session, which hardly sufficed, however, for the adequate presentation of the unusually large number of papers offered. The Council announced the election of the following persons to membership in the Society:—Mr. John B. Faught, Indiana University, Bloomington, Ind.; Mr. Edward B. Fishburne, Waynesboro, Va.; Professor William P. Graham, Syracuse University, Syracuse, N. Y.; Dr. Waldemar Schulz, Cornell University, Ithaca, N. Y.; Dr. Ernest J. Wilczynski, University of California, Berkeley, Cal. Four applications for membership were reported.

An amendment to the Constitution was declared adopted, by which retiring Presidents of the Society continue to serve in the Council for one year. It was announced that the summer meeting of the Society would be held at Columbus, Ohio, on Friday and Saturday, August 25–26, in affiliation with the American Association for the Advancement of Science.

The President then made, in behalf of the Council, an announcement of deep significance for the future of the Society. For nearly a year plans have been under discussion for providing adequate facilities for the publication of the constantly increasing number of valuable original mathematical papers produced in America. The committee appointed at the last summer meeting to consider this question has reported that it is desirable and feasible and in all respects for the best interests of mathematical science in this country that the Society should undertake the periodical publication of Transactions, beginning with January 1, 1900. This is a project which is ardently supported by many of the most prominent members of the Society. The general interest which it has aroused is already stimulating the Society to increased activity, the immediate effect being reflected in the attendance and number of papers presented at this meeting. The Council has appointed a committee, consisting of Professors T. S. Fiske, R. S. Woodward, E. H. Moore, M. Bôcher, and J. Pierpont, to secure the necessary financial guarantees for the proposed undertaking, the success of which is already well assured.

The following papers were presented at the meeting:

(1) Professor M. I. PUPIN: "Electrical oscillations on a loaded conductor."

(2) Professor MAXIME BÔCHER: "An elementary proof that Bessel's functions of the zeroth order have an infinite number of real roots."

(3) Professor J. M. PEIRCE: "Determinants of quaternions."

(4) Professor HENRY TABER: "The chief theorem of the theory of finite continuous groups."

(5) Professor ALEXANDER MACFARLANE: "On the imaginary of geometry."

(6) Professor W. F. OSGOOD: "On a means of generating a function of a real variable whose derivative exists for every value of the argument but is not integrable."

(7) Professor JAMES PIERPONT: "On arithmetizing mathematics."

(8) Professor E. O. LOVETT: "On a certain class of differential invariants."

(9) Dr. VIRGIL SNYDER: "Lines of curvature on annular surfaces having two spherical directrices."

(10) Professor E. W. BROWN: "On the progress of the calculations in the new lunar theory."

(11) Professor M. I. PUPIN: "Lagrange's equations and the principle of equality of action and reaction."

(12) Professor E. B. VAN VLECK: "On the determination of a series of Sturm's functions by the computation of a single determinant."

(13) Dr. G. A. MILLER: "On the primitive groups of degree 17."

(14) Dr. L. E. DICKSON: "Concerning the abelian and hypoabelian groups."

(15) Dr. F. H. SAFFORD: "Surfaces of revolution in the theory of Lamé's products."

(16) Dr. D. F. CAMPBELL: "On linear differential equations of the third and fourth orders in whose solutions exist certain homogeneous relations."

(17) Mr. E. R. HEDRICK: "On three dimensional determinants."

(18) Dr. G. H. LING: "An examination of groups whose orders lie between 1093 and 2000."

(19) Professor A. G. WEBSTER: "Traces illustrating the motion of the top."

In the absence of the authors the papers of Dr. Dickson, Dr. Safford and Dr. Campbell were read by title. Professor Peirce's paper was read by Professor Bôcher, and Dr. Snyder's by Dr. Miller. Mr. Hedrick was introduced by Professor Osgood. Dr. Campbell's paper was offered to the Society through Professor Bôcher. Abstracts of papers not intended for publication in the BULLETIN are given below.

Professor Macfarlane elucidates the theory of geometric imaginaries. In his paper on multiple algebra Professor Cayley identifies the imaginary of algebra with the imaginary of analytical geometry, and says: "In analytical geometry, without seeking for any real representation, we deal with imaginary points, lines, etc., depending on parameters of the form $a + bi$." The object of the present paper is to investigate the real representation of the so called imaginary points, lines, etc. Some authors, *e. g.*, Carr in his Synopsis, give a fictitious representation of the ideal intersection of a straight line and a circle, by drawing the supplementary hyperbola in a plane at right angles to the plane of the given circle; and it is supposed that the $\sqrt{-1}$ denotes the turning of the plane of the supplementary curve into the position at right angles to the original plane. There is no proof or application of the construction. It is not true, as Carr says, that the theory of quaternions removes all imaginariness from the symbol $\sqrt{-1}$. For Hamilton said: "It would be an error to confound geometrical imaginaries

with those square roots of negatives for which the calculus of quaternions supplies from the outset a definite and real representation. M. Maximilien Marie in his representation does not go out of the original plane, but gets his *realized* curve by dropping the $\sqrt{-1}$ from the coördinates in which it appears. In the present investigation it is shown that when the roots of a quadratic equation are real, they are still capable of a true representation in the plane, that is, as complex quantities. They have then the same relation to hyperbolic trigonometry, that the pair of roots, when imaginary, have to circular trigonometry. The fundamental theorem for the complex angle is investigated; and it is shown that the pair of roots, when imaginary, may have a real linear representation, namely, as proportional to the cosine or the sine of a complex angle. The theory is then applied to the imaginary intersection of a straight line and circle, to drawing a tangent to a circle from an internal point, to the imaginary intersection of a straight line and equilateral hyperbola, to the intersection of two circles, to imaginary lines and imaginary circles.

The following is an abstract of Professor Osgood's paper: Let the function $f(x)$ be defined as follows:

$$f(x) = x^2 \sin \frac{1}{x}, \quad \text{when } x \neq 0; \qquad f(0) = 0.$$

Let $a_1, a_2, \cdots$ be the abscissas of maxima of the graph of $f(x)$, so taken that $a_1 > a_2 > \cdots > 0$. Form the function

$$\varphi_n(x) = f(x), \quad \text{when } 0 \leqq x \leqq a_n;$$

$$\varphi_n(x) = f(2a_n - x), \quad \text{when } a_n < x \leqq 2a_n.$$

The graph of this function in the interval $a_n \leqq x \leqq 2a_n$ is the reflection in the ordinate whose abscissa is a_n of the graph of $f(x)$ in the interval $0 \leqq x \leqq a_n$. Finally let

$$\omega_n(x) = \frac{1}{2a_n} \varphi_n(2a_n x), \quad \text{when } 0 \leqq x \leqq 1;$$

$$\omega_n(x) = 0, \quad \text{when } x > 1 \text{ or } x < 0.$$

This function, $\omega_n(x)$, whose graph is of the same *form* as the graph of $\varphi_n(x)$, but drawn to a different *scale*, if $2a_n \neq 1$, has for every value of x, a derivative, $\omega_n'(x)$, which is in general continuous, but in the neighborhood of the points $x = 0$ and $x = 1$ oscillates between limits nearly equal to $+1$ and -1.

Next consider the Cantor's set described in the *American Journal of Mathematics*, vol. 19 (1897), p. 167, §8; BULLETIN, November, 1898, p. 83. Let the set be placed in the interval $0 \leqq x \leqq 1$. Consider any one (the ith) of the 2^{n-1} subintervals (n); its length is denoted by l_n; let the abscissa of the extremity that lies nearer the point $x = 0$ be denoted by $x_n^{(i)}$. I now proceed to "fit the function $\omega_n(x)$ into this interval." I wish this expression to describe the process of forming the function

$$\Omega_n^{(i)}(x) = l_n \omega_n\left(\frac{x - x_n^{(i)}}{l_n}\right),$$

whose graph is of the same form as that of $\omega_n(x)$, but drawn to a different scale. This being done for each one of the intervals (1), (2), ···, a set of functions is obtained by means of which a function

$$F(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} \Omega_n^{(i)}(x)$$

is defined, which has a finite derivative, $F'(x)$, for every value of x. The function $F'(x)$ is finite throughout the whole interval $0 \leqq x \leqq 1$, but near the extremities of the intervals (n) it oscillates between limits nearly equal to $+1$ and -1. These points form a set of positive content, and hence $F'(x)$ cannot be integrated *throughout the whole interval* $(0, 1)$, though it can be integrated throughout certain parts of it.

It remains to "fit the function $F(x)$ into the free intervals" (1), (2), ···, its amplitude being simultaneously diminished. Let

$$F_n^{(i)}(x) = l_n^2 F\left(\frac{x - x_n^{(i)}}{l_n}\right),$$

$$F_1(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} F_n^{(i)}(x);$$

and let $c_n = 10^{-n}$. Then the function $F(x) + c_1 F_1(x)$ is still integrable throughout certain intervals that lie within the former intervals (1), (2), ···. The function $F(x)$ must be "fitted into these intervals," its amplitude being diminished, and a function $F_2(x)$ formed in a manner similar to that in which $F_1(x)$ was just constructed. Proceeding in this way, we obtain a set of functions $F_k(x)$, $(k = 1, 2, \cdots)$, and the function

$$\Phi(x) = F(x) + c_1 F_1(x) + c_2 F_2(x) + \cdots$$

is the function required. It has a derivative for each value of x in the interval (0, 1). This derivative is finite throughout the interval; but it cannot be integrated throughout any portion of the interval.

Professor Lovett's paper constructs a class of $\frac{1}{2}n^2(n+1)$ second order differential invariants in n variables under the general projective group in n variables. These invariants are found in two ways: 1° by elimination, when the finite equations of the group are taken as starting point; 2° by integration, when the infinitesimal transformations alone are given, namely the integration of the complete system of partial differential equations formed by equating to zero the second extensions of the infinitesimal generators of the general projective group. For the values 1, 2, and 3 of n, the above system reduces to Schwarz's derivative, the invariants of Goursat, and those of Painlevé, respectively.

The results of Dr. Snyder's paper are summarized as follows: When the coördinates of the spheres which generate an annular surface satisfy two linear relations, all of the lines of curvature of the second system lie on spheres; these spheres can be determined by rational operations. The points of intersection of each such sphere with the circles of the first system lie in involution. The center of this involution is a plane curve, which touches a fixed circle in $2n$ points. All the lines of curvature pass through the "pinchpoints" and touch each other. Every such surface has one plane line of curvature of the second system; if it has two, all become plane and these planes all belong to the same axial pencil. Surfaces of revolution make one type of such surfaces. When the two directrices coincide, all of the spheres pass through a fixed circle; this is a line of curvature on the surface. In case the double directrix is a plane, a torsal line lies on the surface, which is a line of parabolic curvature. All the other planes containing lines of curvature pass through this line. The involution is now degraded. If the planes of a developable touch a fixed sphere, all its lines of curvature are rational; they lie on a family of spheres concentric with the fixed one. Cones form a particular example. The spherical line of curvature on the point locus of any complex is determined. The minimum line of curvature is determined for every surface.

There are five primitive groups of degree 17 that are in-

cluded in the metacyclic group of this degree. In Jordan's enumeration of the primitive groups of degree 17 (*Comptes Rendus*, vol. 75, p. 1757) only one additional group besides the alternating and the symmetric is given. Dr. Miller finds that there are at least three such groups whose orders are 17.240, 17.480, 17.960. The first of these is a simple group and is included in each of the other two.

Dr. Dickson's paper gives the results of an investigation made a year ago which led to two new definitions of the abelian and the two hypoabelian linear groups. For the first definition, the author employs the conception of the group of isomorphisms of an *abstract* group, which was arrived at by Moore and Hölder independently. Consider the abstract group F generated by the operators θ, $A_1, \ldots, A_m, B_1, \ldots, B_m$ subject to the complete set of generational relations

$$\theta^p = A_i^p = B_i^p = 1, \quad \theta A_i = A_i\theta, \quad \theta B_i = B_i\theta \quad (i = 1, \cdots, m)$$

$$A_iA_j = A_jA_i, \quad B_iB_j = B_jB_i \qquad (i, j = 1, \cdots, m)$$

$$A_iB_i = \theta\, B_iA_i, \quad A_iB_j = B_jA_i \quad (i, j = 1, \cdots, m;\ i \neq j).$$

Limiting the discussion to the case $p =$ prime, every operator of F can be expressed in a single way in the form

$$\theta^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

the exponents t, x_i, y_i being taken modulo p. Introducing in the most general way a new set of generators θ', A_i', B_i' $(i = 1, \cdots, m)$ satisfying the above relations and capable of generating the entire group F, we obtain as the group of isomorphisms of F into itself a group meriedrically isomorphic with a $2m$-ary linear homogeneous group modulo p. For $p > 2$, the latter is the general abelian group; for $p = 2$, it is the first hypoabelian group.

Extending the group F by an operator J commutative with every operator of F and such that $J^p = \theta$, we obtain a group F_1 whose general operator can be expressed in a single way in the form

$$J^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

the exponents x_i, y_i being taken modulo p, while t is taken modulo p^2. The group of isomorphisms of F_1 is holoedrically isomorphic with the general abelian group on $2m$ indices

taken modulo p. Similarly, a third abstract group F' is defined, whose group of isomorphisms is meriedrically isomorphic with the total second hypoabelian group if $p = 2$, and to a subgroup of the general abelian group if $p > 2$. Utilizing certain developments of Jordan on solvable groups (Traité, §§ 561 and 566), the author gives a concrete representation of the above groups F, F_1, and F' as elementary groups of linear homogeneous substitutions on p^m variables. The above results are then generalized so that we obtain, as groups of isomorphisms of certain very elementary groups, the general abelian and the first hypoabelian groups on $2m$ indices in the Galois field of order p^n.

The second definition of the general abelian linear group results from a simple application of the very fruitful concept of "the compound groups of a given linear homogeneous group" developed by the author in recent articles in the BULLETIN and in the *Proceedings of the London Mathematical Society*. THEOREM:—*The general abelian group on $2m$ indices is the largest $2m$-ary linear homogeneous group whose second compound possesses a certain linear invariant.*

The author then proves that the simple groups of order

$$\tfrac{1}{2}(p^{4n} - 1)\ (p^{2n} - 1)p^{4n}$$

derived from the decomposition of the abelian group on 4 indices in the $GF[p^n]$ and those derived from the decomposition of the orthogonal group on 5 indices in the $GF[p^n]$ are holoedrically isomorphic.

Wangerin* has treated the problem of obtaining the most general orthogonal surfaces of revolution such that, if Laplace's equation be written in coördinates corresponding to these surfaces, a solution may be obtained in the form of a Lamé's product with an extraneous factor. His result is that the meridian curves and the surfaces are of the fourth degree. Haentzschel† claims to have obtained surfaces of the thirty-second degree and denies the generality of Wangerin's result. In Dr. Safford's paper Haentzschel's surfaces are shown to be of the sixteenth degree only, and are resolved into four surfaces of the fourth degree. Thus Wangerin's result is proved to be correct and Haentzschel's criticism unfounded.

In the *Acta Mathematica*, volume 14 (1890), is a memoir by Brioschi in which he considers in a number of special

* *Berl. Monatsber.*, Feb., 1878.

† E. Haentschel, "Studien über die Reduction der Potentialgleichung, auf gewöhnliche Differentialgleichungen," Berlin, Reimer, 1893.

cases the question of homogeneous relations of degree m in the solutions of a linear differential equation of order n. He assumes that a relation $f_1(y_1, y_2, \cdots, y_n) = 0$ of degree m exists and then shows that a certain invariant must vanish. Instead of a relation, Dr. Campbell takes a form, $u(x) = f(y_1, y_2, \cdots, y_n)$ with arbitrary coefficients. Then instead of Brioschi's two equations of the form*

$$\varphi(\lambda) = 0, \quad \psi(\lambda) = 0,$$

we have,

$$\varphi(\lambda) = \chi(u), \quad \psi(\lambda) = \varkappa(u),$$

where $\varphi(\lambda) = 0$, $\psi(\lambda) = 0$, $\chi(u) = 0$, $\varkappa(u) = 0$, would be linear differential equations in λ and u respectively. The following theorems can be established without much difficulty: 1° If ν and only ν linearly independent relations of the first degree exist among g functions, these g functions are the solutions of a linear differential equation of order $g - \nu$. 2° In the cases mentioned below, λ or $f(y_1', y_2', \cdots, y_n')$ cannot vanish identically. By means of these theorems Dr. Campbell finds, by an extension of Brioschi's method: (*a*) In case $m = 2$, $n = 3$, there cannot exist more than one relation. A necessary and sufficient condition that one relation exists is that the invariant a_3 vanishes. (*b*) In case $m = 3$, $n = 3$, there cannot exist more than three, or two and only two, linearly independent relations. A necessary and sufficient condition for the existence of three linearly independent relations is that the invariant a_3 vanishes; and for one and only one relation, that the invariant Ω_{21} vanishes and a_3 does not. (*c*) In case $m = 2$, $n = 4$, there cannot exist more than three linearly independent relations. A necessary and sufficient condition for the existence of three linearly independent relations is that the invariants a_3 and a_4 vanish; of two and only two linearly independent relations, that the invariants b_8 and Ω_{21} vanish and neither a_3 nor a_4 vanishes; of one and only one relation, that the invariant Ω_{21} vanishes and b_8 does not vanish. In the course of the work appears the linear differential equation of which u is the general solution, in each of the above cases. Brioschi transforms the equation to a standard form when a relation exists. By means of this form, in each of the cases (*a*) and (*c*). the equation can be solved by solving equations of lower order. This has long since been known in case (*a*). In case (*c*) it can be shown without much difficulty that the equation can be solved by solving one linear differential equation of

* For the definitions of functions here used see the memoir above referred to.

the third order and either one or two linear differential equations of the second order according as the C of the standard form is or is not zero.

Mr. Hedrick's paper will appear in the *Annals of Mathematics*. In 1861, de Gasparis extended the determinant notation to three dimensions. He was followed by several writers,* notably Zehfuss, Scott, and Gegenbauer. The latter works in n dimensions; but simplicity is gained in notation without any real loss in generality by working in three dimensions, and in this way also the aid of the geometric picture of the cubical array is secured. Mr. Hedrick employs a new definition of a cubic determinant, consisting in a development into ordinary determinants, to prove a number of elementary theorems closely corresponding to the theorems in ordinary determinants, except for an essential lack of symmetry, with respect to one of the three sets of mutually perpendicular lines in the cubic determinant. A theorem for the multiplication of a cubic determinant by an ordinary determinant is proved; and is used to show that certain cubic determinants, somewhat analogous to the ordinary Hessian, are simultaneous covariants of a system of forms. The extension of this theorem to n dimensions gives a very general simultaneous covariant of a system of forms.

Dr. Ling continues the search for simple groups. Commencing where the subject was left by Professor Burnside's paper presented to the London Mathematical Society in 1895, the search is extended up to groups whose order is 2000. All but four of the groups whose orders lie between 1,093 and 2,000 have been examined, no new simple groups being found. The methods are almost entirely the same as those which have been used in the examination of groups of degree lower than 1,093. No result is found regarding the existence of simple groups of orders 1440, 1512, 1680, or 1800.

Professor Webster exhibited a large number of curves traced by the motion of a rotating top.

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* See Encykl. der Math. Wiss., vol. 1, no. 1. p. 45.

DETERMINANTS OF QUATERNIONS.

BY PROFESSOR JAMES MILLS PEIRCE.

(Abstract, read before the American Mathematical Society at the Meeting of February 25, 1899.)

I ADOPT as the basis of the following discussion the convention that the order of factors in every term of a determinant is the same as the order of columns in the matrix; while the order of ranks in each term fixes the sign of the term, according to the usual rule for determinants of scalars. It follows that, in quaternions, columns and ranks are not interchangeable; that the transposition of ranks causes the same changes of sign as in scalar algebra; and that a determinant which has two equal ranks vanishes; but that the position of a column is, in general, fixed; and that a determinant may have two or more equal columns without vanishing. If, however, a determinant has a scalar column, such column may be displaced with the same changes of sign as in scalar algebra; and if two scalar columns of a matrix are equal, the determinant vanishes.

The usual development of a determinant as a sum of products of constituents from any row into corresponding minors can be employed only for the first and last columns of a determinant of quaternions, or for a scalar column or rank; but a more general formula, preserving the order of factors in each term, may be employed for *any* column or rank. The addition theorem holds for determinants of quaternions; but the multiplication theorem does not hold, since its proof involves the commutative property of multiplication. The latter theorem may be used, however, when one of the factors is a determinant of scalars; a case which may often occur in applications.

A *Repeating Determinant* is defined as a determinant of which every column is either equal or conjugate to an assumed standard column; and a *Resultant* of several quaternions, as a repeating determinant found from those quaternions, and divided by the number of its terms, that is by the factorial of its order. A single quaternion has two resultants, which are equal respectively to the quaternion and to its conjugate; and n quaternions have 2^n resultants, among which we distinguish two *uniform* resultants, of which the columns are all equal, and two *alternating* result-

ants, of which the columns are alternately normal and conjugate.

Since all scalars are self-conjugate, the two resultants of a single scalar are equal; so that one scalar has but one resultant, which is the scalar itself. Since a determinant having two equal scalar columns vanishes, *every resultant of two or more scalars is identically equal to zero.*

Since the conjugate of a vector is the same as its negative, the different resultants of a given system of vectors are all equal except in sign. Hence, in the case of vectors, we may confine ourselves to *uniform* resultants. We may now compute resultants of the successive orders, either by experiment, or (more instructively) by the aid of general rules of the calculus of quaternions. If the resultant of a system of quantities be denoted by writing the letter R. before the principal diagonal term of the determinant, we have, for the resultants of vectors,

$$\mathrm{R}.\alpha_1\alpha_2 = \tfrac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1) = \mathrm{V}\alpha_1\alpha_2,$$

$$\mathrm{R}.\alpha_1\alpha_2\alpha_3 = \tfrac{1}{6}(\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_3\alpha_2 + \alpha_2\alpha_3\alpha_1 - \alpha_2\alpha_1\alpha_3 + \alpha_3\alpha_1\alpha_2 - \alpha_3\alpha_2\alpha_1) = \mathrm{S}\,\alpha_1\alpha_2\alpha_3,$$

$$\mathrm{R}.\alpha_1\alpha_2\alpha_3\alpha_4 = \tfrac{1}{24}(\alpha_1\alpha_2\alpha_3\alpha_4 - \alpha_1\alpha_2\alpha_4\alpha_3 + \text{etc.}) = 0;$$

whence it follows, by the development in minors, that *every resultant of more than three vectors vanishes identically.*

The relation of the forms $\mathrm{V}\alpha_1\alpha_2$ and $\mathrm{S}\alpha_1\alpha_2\alpha_3$ to certain scalar determinant forms is well known. But it seems not to have been observed that they are *themselves determinants.*

To find the resultants of a system of n quaternions, we may break up each determinant into its scalar and vector parts, and then, by the addition theorem, resolve the required resultant into the sum of 2^n determinant parts. But, since every determinant vanishes, which has two equal scalar columns, there will remain only n determinants having one scalar column each, which are all equal save in sign, and *one* determinant of vectors. Thus, the computation is reduced to that of resultants of vectors of the orders n and $n-1$.

We may confine ourselves here to *alternating* resultants, which are found to be the best adapted to use in theorems relating to linear equations. We shall find it convenient to denote the conjugate of a quaternion by a *dash* over the letter which denotes the quaternion; writing, for example,

$$\bar{p} = \mathrm{K}\,p.$$

Using the method of computation indicated above and denoting by a and a respectively the scalar and vector parts of a quaternion p, we now easily obtain the following results:

$$\mathrm{R}.p_1p_2 = a_2a_1 - a_1a_2 - \mathrm{V}a_1a_2,$$

$$\mathrm{R}.p_1p_2p_3 = -a_1\mathrm{V}a_2a_3 - a_2\mathrm{V}a_3a_1 - a_3\mathrm{V}a_1a_2 - \mathrm{S}a_1a_2a_3,$$

$$\mathrm{R}.\bar{p}_1p_2p_3p_4 = a_1\mathrm{S}a_2a_3a_4 - a_2\mathrm{S}a_1a_3a_4 + a_3\mathrm{S}a_1a_2a_4 - a_4\mathrm{S}a_1a_2a_3,$$

$$\mathrm{R}.\bar{p}_1\bar{p}_2p_3p_4p_5 = 0;$$

whence it may be proved that *every resultant of more than four quaternions vanishes identically.* It will be found that a *uniform* resultant vanishes even for four quaternions.

It may now be shown that the vanishing of an alternating resultant of a system of quaternions is the necessary and sufficient condition of the existence of a linear equation with scalar coefficients between those quaternions; but the vanishing of a *uniform* resultant is not in all cases a *sufficient*, though always a *necessary*, condition of the existence of the equation. In the case of a system of *vectors*, however, the vanishing of *any* resultant involves the vanishing of all the resultants. It is well known that $\mathrm{V}\alpha\beta = 0$ and $\mathrm{S}\alpha\beta\gamma = 0$ are respectively the conditions of the existence of equations of the forms

$$x\alpha + y\beta = 0, \quad \text{and} \quad x\alpha + y\beta + z\gamma = 0;$$

and that *any four* vectors satisfy a linear equation. The conditions for quaternions are given in another (somewhat empirical) form by Hamilton. But their relation to the theory of determinants is not indicated and this seems to the writer to give the true key to the subject of linear equations.

The vanishing of a resultant of several quaternions is not only the *condition* of the existence of a linear equation, but it is *itself equivalent to that equation*. Thus the condition $\mathrm{R}.a_1a_2a_3 = 0$, may be written

$$\tfrac{1}{3}(a_1\mathrm{R}.a_2a_3 - a_2\mathrm{R}.a_1a_3 + a_3\mathrm{R}.a_1a_2) = 0,$$

or (removing the factor $\frac{1}{3}$),

$$a_1\mathrm{V}a_2a_3 + a_2\mathrm{V}a_3a_1 + a_3\mathrm{V}a_1a_2 = 0;$$

where, when $\mathrm{R}.a_1a_2a_3 = 0$, the vector coefficients of a_1, a_2, a_3, have scalar ratios to each other, so that by removing a common vector factor, the desired linear equation with scalar coefficients is at once obtained. So (or more directly) in other cases.

ST. PETERSBURG, *January*, 1899.

THE LARGEST LINEAR HOMOGENEOUS GROUP WITH AN INVARIANT PFAFFIAN.

BY DR. L. E. DICKSON.

(Read before the American Mathematical Society at the Meeting of October 29, 1898.)

1. In the December number of the BULLETIN (pp. 120–135) I have shown that the second compound of the general $2m$-ary linear homogeneous group is a linear group in $C_{2m,2} \equiv m(2m-1)$ variables which leaves invariant the Pfaffian

$$F \equiv [1, 2, \cdots, 2m].$$

Denoting the variables as follows:

$$Y_{ij} \equiv -Y_{ji} \qquad (i, j = 1, \cdots, 2m\,;\ i \neq j), \tag{1}$$

the second compound was proved to contain exactly $(2m)^2$ linearly independent infinitesimal transformations

$$\sum_{r \neq s,\, t}^{r=1,\cdots,2m} Y_{rt} \frac{\partial f}{\partial Y_{rs}} \delta t. \qquad (t, s = 1, \cdots, 2m). \tag{2}$$

The object of the present note is to prove that the largest linear homogeneous group G in the $m(2m-1)$ variables (1) which leaves invariant the Pfaffian F contains only the $(2m)^2$ linearly independent transformations (2).

2. Let the general infinitesimal transformation of the group G be as follows:

$$\delta Y_{ij} = \sum_{k \neq i}^{k=1,\cdots,2m} a_{kl}^{ij} Y_{kl} \delta t \qquad (i, j = 1, \cdots, 2m\,;\ i \neq j), \tag{3}$$

where, on account of (1), we may suppose

$$a_{kl}^{ij} = -a_{lk}^{ij} = +a_{ik}^{jl}. \tag{4}$$

The condition that (3) shall multiply F by a constant $c\delta t$ is as follows:

$$\sum_{i,j,k,l} \frac{\partial F}{\partial Y_{ij}} a_{kl}^{ij} Y_{kl} = cF. \tag{5}$$

Now

$$\frac{\partial F}{\partial Y_{ij}} = \frac{\partial}{\partial Y_{ij}}\{(-1)^{i+j-1}[ij\,1\,2\cdots i-1\,i+1\cdots j-1\,j+1\cdots 2m]$$

$$= (-1)^{i+j-1}[1\,2\cdots i-1\,i+1\cdots j-1\,j+1\cdots 2m].$$

Comparing the coefficients of the terms in (5) of the type

$$(-1)^{\sigma} Y_{i_1 i_2} Y_{i_3 i_4}\cdots Y_{i_{2m-1} i_{2m}},$$

where $i_1, i_2, \cdots, i_{2m}$ is a permutation of $1, 2, \cdots, 2m$ and where σ denotes the number of transpositions giving that permutation, we obtain the conditions

$$a^{i_1 i_2}_{i_1 i_2} + a^{i_3 i_4}_{i_3 i_4} + \cdots + a^{i_{2m-1} i_{2m}}_{i_{2m-1} i_{2m}} = c. \tag{6}$$

Comparing the coefficients of the terms,

$$Y^2_{i_3 i_4} Y_{i_5 i_6}\cdots Y_{i_{2m-1} i_{2m}}$$

we obtain the conditions

$$a^{i_1 i_2}_{i_3 i_4} = 0 \qquad (i_3 \text{ and } i_4 \neq i_1 \text{ or } i_2). \tag{7}$$

Comparing the coefficients of the terms

$$Y_{i_1 i_3} Y_{i_2 i_4}\cdots Y_{i_{2m-1} i_{2m}},$$

we find

$$a^{i_1 i_2}_{i_1 i_3} - a^{i_4 i_2}_{i_4 i_3} = 0 \tag{8}$$

(i_1, i_2, i_3, i_4, being any four different integers $\leqq 2m$).

We may now obtain a complete set of linearly independent infinitesimal transformations (3), which leave F invariant. According as every $a^{i_1 i_2}_{i_3 i_4}$ is zero, or not every such a is zero, we obtain two independent types of transformations (3), which together form the desired complete set. We consider the two types in succession:

(*a*) If any $a^{rs}_{rt} \neq 0$, say $= 1$, where r, s, t are distinct integers $\leqq 2m$, then by (8) we have

$$a^{rs}_{rt} = 1 \quad (r = 1, \cdots, 2m\,;\ r \neq s, t).$$

Setting every other $a = 0$, we obtain a set of solutions of (6), (7), (8), for which

$$\begin{cases} \delta Y_{rs} = Y_{rt}\delta t & (r = 1, \cdots, 2m\,;\ r \neq s, t) \\ \delta Y_{ij} = 0 & (i, j = 1, \cdots, 2m\,;\ i \neq r). \end{cases}$$

We thus obtain the $2m\,(2m-1)$ infinitesimal transformations (included in the formula (2))

$$(9) \qquad \sum_{r+s,\,t}^{r=1,\cdots,2m} Y_{rt}\frac{\partial f}{\partial Y_{rs}}\delta t \qquad \left(\begin{matrix} s,t=1,\cdots,2m \\ s+t \end{matrix}\right),$$

which are therefore linearly independent.

(*b*) If next

$$a_{rt}^{rs}=0 \quad (r,s,t=1,\cdots,2m\,;\,r+s,+t),$$

the general transformation (3) becomes

$$\delta Y_{ij}=a_{ij}^{ij}Y_{ij}\delta t \qquad (i,j=1,\cdots,2m),$$

where the a_{ij}^{ij} are subject to the conditions (6).

Writing for brevity [see (4)],

$$a_{ij}^{ij}\equiv(ij)=(ji),$$

these conditions (6) become

$$(6) \qquad (i_1i_2)+(i_3i_4)+\cdots+(i_{2m-1}i_{2m})=c.$$

We obtain at once the following $2m$ sets of solutions of these equations, each set being given by one value of l chosen from $1, 2, \cdots, 2m$;

$$(10)\begin{cases}(l1)=(l2)=\cdots=(ll-1)=(ll+1)=\cdots=(l2m)=c \\ (ij)=0 \qquad\qquad [i,j=1,\cdots,2m\,;\;i+l].\end{cases}$$

These sets of solutions of the equations (6), (7), (8) give rise to the following $2m$ infinitesimal transformations:

$$(11) \qquad A_{il}\equiv\sum_{j+l}^{j=1,\cdots,2m}Y_{ij}\frac{\partial f}{\partial Y_{ij}} \qquad (l=1,\cdots,2m).$$

These transformations are linearly independent if $m\geqq 2$. Indeed, if

$$\sum_{l=1}^{2m}k_lA_{il}=0,$$

upon equating the coefficients of $\dfrac{\partial f}{\partial Y_{rs}}$ in the two members, we have

$$k_r+k_s=0 \qquad (r,s=1,\cdots,2m\,;\,r+s).$$

Hence, if $m\geqq 2$, $k_l=0\,(l=1,\cdots,2m)$.

The transformations (9) and (11) make up the $(2m)^2$ linearly independent transformations (2). It follows from the theorem of the next paragraph that there do not exist more than $2m$ linearly independent transformations of the type (b). We will then have proved the following theorem:

The largest linear homogeneous group in $C_{2m,2}$ variables leaving invariant the Pfaffian $[1, 2, \cdots, 2m]$ is identical with the second compound of the general m-ary linear homogeneous group.

3. THEOREM. *The $m(2m-1)$ quantities*

$$(ij) \equiv (ji) \qquad [i, j = 1, 2, \cdots, 2m,\ i \neq j]$$

satisfying the $1 \cdot 3 \cdot 5 \cdots (2m-3)(2m-1)$ equations

$$[E_{2m}] \qquad (i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-1} i_{2m}) = c_{2m}$$

can all be expressed in terms of certain $2m$ of the (ij), for example,

$$[Q_{2m}] \begin{cases} (1\,2),\ (3\,4),\ (5\,6),\ \cdots,\ (2l-1\ 2l),\ \cdots,\ (2m-1\ 2m); \\ (2\,3);\ (2\,4),\ (4\,6),\ \cdots,\ (2l-2\ 2l),\ \cdots,\ (2m-2\ 2m), \end{cases}$$

but not in terms of fewer than $2m$ of them if $m > 1$.

The last part of the theorem follows from the linear independence of the $2m$ infinitesimal transformation of type (b) above.

The first part of the theorem will be proved by induction. For $m = 2$, it is evident; for the equations $[E_4]$ are as follows:

$$(12) + (34) = (13) + (24) = (14) + (23) = c_4.$$

Supposing the first part of the theorem to be true for a given value of $2m$, we can prove it true for the next value $2(m+1)$. Indeed, applying this hypothesis to certain equations of the set $[E_{2m+2}]$, viz.:

$$(i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-1}\, i_{2m}) = c_{2m+2} - (2m+1\ \ 2m+2),$$

where $i_1, i_2, \cdots, i_{2m}$ is a permutation of $1, 2, \cdots, 2m$, it follows that the quantities

$$(ij) \qquad [i, j = 1, 2, \cdots, 2m;\ i \neq j]$$

can be expressed in terms of the quantities Q_{2m} and that c_{2m+2} is expressible in terms of the quantities Q_{2m} together with $(2m+1\ 2m+2)$.

Consider next the equations of the set $[E_{2m+2}]$

$$(i_1 i_2) + (i_3 i_4) + \cdots + (i_{2m-3}\, i_{2m-2}) + (j\ 2m+1) + (2m\ 2m+2) = c_{2m+3},$$

where $i_1, i_2, \cdots, i_{2m-2}, j$ form a permutation of $1, 2, \cdots, 2m-1$. It follows that every

$$(j\ 2m+1) \qquad [j = 1, 2, \cdots, 2m-1]$$

is expressible in terms of the Q_{2m}, c_{2m+2} and $(2m\ 2m+2)$ and hence in terms of the Q_{2m+2}.
From the equation

$$(1\ 2) + (3\ 4) + \cdots + (2m-5\ 2m-4) + (2m-3\ 2m)$$
$$+ (2m-2\ 2m+1) + (2m-1\ 2m+2) = c_{2m+2}$$

we have $(2m-1\ 2m+2)$ expressed in terms of the quantities Q_{2m+2} (by using our earlier results). Hence, from the equation

$$(1\ 2) + (3\ 4) + \cdots + (2m-3\ 2m-2) + (2m\ 2m+1)$$
$$+ (2m-1\ 2m+2) = c_{2m+2},$$

we obtain $(2m\ 2m+1)$ expressed in terms of the Q_{2m+2}. We have, therefore, every

$$(j\ 2m+1) \quad [j = 1, 2, \cdots, 2m+2]$$

expressed in terms of the Q_{2m+2}.

Finally, from the equations

$$(i_1 i_2) + \cdots + (i_{2m-1} i_{2m}) + (j\ 2m+2)$$
$$+ (2m-1\ 2m+1) = c_{2m+2},$$

where $j, i_1, i_2, \cdots, i_{2m}$ form a permutation of $1, 2, \cdots, 2m-2$, $2m$, we are able to express

$$(j\ 2m+2) \quad [j = 1, 2, \cdots, 2m-2, 2m]$$

in terms of the Q_{2m+2}.

Combining our results, we find that every

$$(i\,j) \quad [i, j = 1, 2, \cdots, 2m+2; i \neq j]$$

is expressible in terms of the quantities Q_{2m+2}.

UNIVERSITY OF CALIFORNIA,
September 23, 1898.

ASYMPTOTIC LINES ON RULED SURFACES HAVING TWO RECTILINEAR DIRECTRICES.

BY DR. VIRGIL SNYDER.

(Read, in part, before the American Mathematical Society, at the Fifth Summer Meeting, August 19, 1898, and at the Annual Meeting, December 29, 1898.)

LET l be a generator of a non-developable ruled surface S, and let π_s be the tangent plane to S at a point P on l. As P moves along l, π_s will turn about l in such a way that the range (P) and the axial pencil (π_s) are homographic (Chasles's correlation).

Again, let l be a line of a linear complex C, and let π_c be the polar plane in C of a point P on l. As P moves along l, π_c will turn about l in such a way that the range (P) and the axial pencil (π_c) are homographic (normal correlation).

The two axial pencils (π_s) and (π_c), being homographic with a common range (P), are projective with each other when l belongs to S and to C. These two projective pencils have two self-corresponding planes, π_1 and π_2, such that the point of tangency and pole coincide; let the corresponding points be P_1 and P_2.

If all the generators l of S belong to C, there will be two points on each, such that the tangent plane and the polar plane coincide. The locus of these points will be a curve traced on the surface, called the *complex curve*.

Let P' be a point on the curve contiguous to P, then PP' is a line of C, hence all the tangents to the curve belong to the complex C. The complex curve is an asymptotic line on the surface, because all its osculating planes are also tangent planes, a characteristic property of the asymptotic lines. This theorem may be stated as follows: *Every ruled surface contained in a linear complex has an asymptotic line, all of whose tangents belong to the complex.*

When the surface is algebraic, the order of this asymptotic line can be easily determined.* Consider any plane α and let it cut the curve in a point K; the polar plane of K with regard to C, which is also the tangent plane to the surface at K, will pass through A, the pole of the plane α. But every line of this polar plane $\varkappa$ of K is a tangent to the sur-

* Picard, "Mémoire sur une application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches." *Annales de l'École Normale*, 1877.

face, hence KA is tangent at K to the section of the surface made by α.

Conversely, let K be the point of contact of a tangent to the curve of this section from the pole A. The plane passed through KA and the generator through A will be tangent to the surface at A. Moreover, K is the pole of this plane, because KA and the generator through K are both lines of the complex, hence K is a point of the complex curve, and the points in which the plane α cuts this curve are the points of contact of the tangents drawn from the pole A of α to the curve of the section which α cuts from the surface S. Now when S is algebraic it follows that: *The degree of the complex curve is the same as the class of any plane section of the surface.*

It was shown by Clebsch* that all the asymptotic lines of a ruled surface can be found by means of integrations, provided one such line be given. It was shown by Bonnet† that the differential equation of the asymptotic lines of ruled surfaces is a Riccati equation; this equation possesses the property that the anharmonic ratio of four solutions is constant (theorem of Paul Serret). Hence, if three such asymptotic lines are known, all the others can be found without an integration. As the complex curve cuts each generator twice, it counts for two asymptotic lines; an integration will determine a third, hence: *All the asymptotic lines of a ruled surface contained in a linear complex can be determined by a single integration.*

The complex curve of an algebraic ruled surface is itself algebraic, but the curves resulting from the integration of an algebraic function are usually transcendental, and in that case all the asymptotic lines except the complex curves are transcendental.

By means of these theorems Picard (*l. c.*) succeeded in determining all the ruled surfaces having algebraic asymptotic lines. Let Q be any algebraic curve, and C a linear complex. Through each point of q and in its osculating plane π lies one line l of C. This line will generate a ruled surface which is contained in C and has q for an asymptotic line; for, let π be the tangent plane to the surface at P on l; the tangents to the curve lie on the surface, hence the osculating plane of every point of q is tangent plane to the surface, *i. e.*, q is an asymptotic line. The tangents to q do

* "Ueber die Curven der Haupttangenten bei windschiefen Flächen," *Crelle's Journal*, vol. 68.

† "Théorie générale des surfaces," *Journal de l'École Polytechnique*, vol. 32.

not belong to C, and since S is contained in C, there are two more asymptotic lines determined as above. The surface now has three algebraic asymptotic lines; hence, from the general property of Riccati's equation which the asymptotic lines of the ruled surface satisfy, all the others are algebraic, and can be at once expressed without an integration.

Belonging to a given curve q there are ∞^5 surfaces which have algebraic asymptotic lines, and which touch each other along q. All ruled surfaces having algebraic asymptotic lines can be determined in this way.

Analytically, one may proceed as follows: let a curve q be defined in terms of a variable μ,

$$x = \varphi_1(\mu), \quad y = \varphi_2(\mu), \quad z = \varphi_3(\mu), \quad w = \varphi_4(\mu);$$

its osculating plane at a given point x, y, z, w is defined by the equation

$$(x, \varphi_2(\mu), \varphi_3'(\mu), \varphi_4''(\mu)) = 0.$$

The polar plane of the same point with regard to a given complex C is of the form

$$x(A\varphi_1 + B\varphi_2 + \cdots) + y(\cdots) + z(\cdots) + w(\cdots) = 0.$$

These two equations define a line of the complex in the pencil (μ,π). The ruled surface is the μ eliminant of these two equations. The succeeding steps are similar to those taken in the preceding case.

Application to Surfaces Having Two Rectilinear Directrices.

(*a*) *Directrices Distinct.*—Let $x = 0$, $y = 0$ be an m-fold line, and $z = 0$, $w = 0$ an n-fold line of a ruled surface; the equation of the surface will then be the expression of an $[m,n]$ correspondence between $\frac{z}{w}$ and $\frac{y}{x}$, namely:

$$(1) \qquad \varphi \equiv z^n {}_1u + z^{n-1} w {}_2u + \cdots + w^n {}_{n+1}u = 0,$$

in which ${}_iu$ is a binary quantic in x,y of degree m.

This surface is contained in the congruence whose directrices are x, y and z, w, which is expressed by the pencil of coaxial complexes

$$(2) \qquad xy' - x'y = k(zw' - z'w),$$

and consequently all the asymptotic lines are algebraic, and

the inflexional tangents are distributed among the complexes of this pencil.

The tangent plane to (1) at the point (x', y', z', w') can be written in the form

$$(3) \quad x' \frac{\partial \varphi}{\partial x'} \cdot \frac{x}{x'} + y' \frac{\partial \varphi}{\partial y'} \cdot \frac{y}{y'} + z' \frac{\partial \varphi}{\partial z'} \cdot \frac{z}{z'} + w' \frac{\partial \varphi}{\partial w'} \cdot \frac{w}{w'} = 0;$$

but $$x' \frac{\partial \varphi}{\partial x'} + y' \frac{\partial \varphi}{\partial y'} = 0, \quad z' \frac{\partial \varphi}{\partial z'} + w' \frac{\partial \varphi}{\partial w'} = 0,$$

(4), hence $$w' \frac{\partial \varphi}{\partial x'} \cdot (xy' - x'y) + y' \frac{\partial \varphi}{\partial z'} (zw' - z'w) = 0.$$

Eliminating $xy' - x'y$ and $zw' - z'w$ between (1) and (4) there results

$$(5) \quad y \frac{\partial \varphi}{\partial z} + kw \frac{\partial \varphi}{\partial x} = 0,$$

which, together with (1), defines the asymptotic lines.

Consider the plane $\lambda x = y$ through the m-fold line; substitute this value of y in (5), and let $\mu w = z$, then

$$(6) \quad \lambda x^2 \frac{\partial \varphi}{\partial z}(\lambda, \mu) + kw^2 \frac{\partial \varphi}{\partial x}(\lambda, \mu) = 0.$$

From (6) the following properties can be immediately deduced:

I. When $\varphi = 0$, $\dfrac{\partial \varphi}{\partial x} = 0$, then $x = 0$ for all values of k; but these equations define those values of $\dfrac{z}{w}$ for which two generators issuing from the same point of the other multiple line coincide, *i. e.*, a *pinch point*, on the line $x = 0$, $y = 0$. Similarly $w = 0$ when $\dfrac{\partial \varphi}{\partial z} = 0$, $\varphi = 0$, which define the pinch points on the other multiple line; hence: *Every asymptotic line passes through all of the pinch points.*

II. Equation (6) together with $x = 0$, $w = 0$ defines an involution; hence: *The points of intersection of the asymptotic lines define an involution on every generator, with the points in which it cuts the multiple lines for foci.*

III. By changing the sign of k, real and imaginary points of intersection of an asymptotic line with a given generator interchange; hence: *No generator is cut by asymptotic lines belonging to both positive and negative complexes.*

It follows from these theorems that the cuspidal generators at the pinch points are inflexional tangents to all the asymptotic lines, as the two points of intersection approach the pinch point symmetrically when x approaches the pinch point.

The degree of these curves is, by Picard's theorem, $2mn - 2\delta$, in which δ is the number of double generators; it can also be obtained by counting the points of intersec-

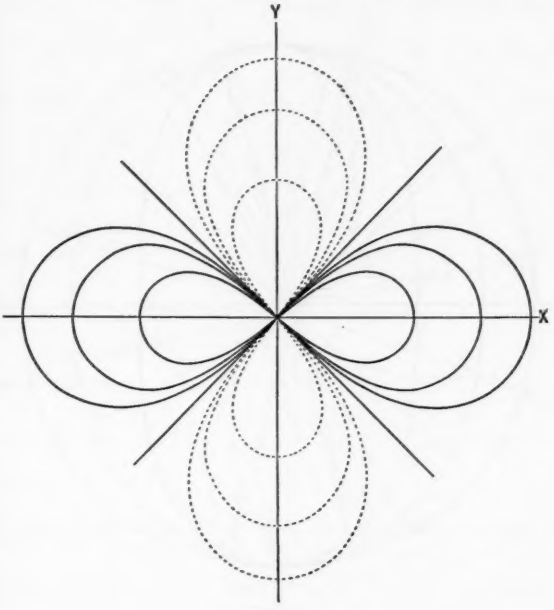

FIG. 1.

tion of any given curve with the plane $\lambda x = y$. This plane passes through the $2n(m-1)$ pinch points on the line xy and each of the n generators contained in the plane has two further points of intersection with the curve, but every double generator diminishes the number of pinch points by 2, hence the number of points is $2mn - 2\delta$, as before. In particular, if $\delta = (n-1)(m-1)$, the surface becomes unicursal and the order of the curves reduces to $2m + 2n - 2$,

as was obtained by Cremona.* The method employed by Cremona will only apply when the surface is unicursal.

No modification is to be made for double generators; the corresponding factors will always divide out.

For the cubic surface [2, 1] the asymptotic lines are of order 4, and pass through the two pinch points on the double line. When the coördinate axes are rectangular

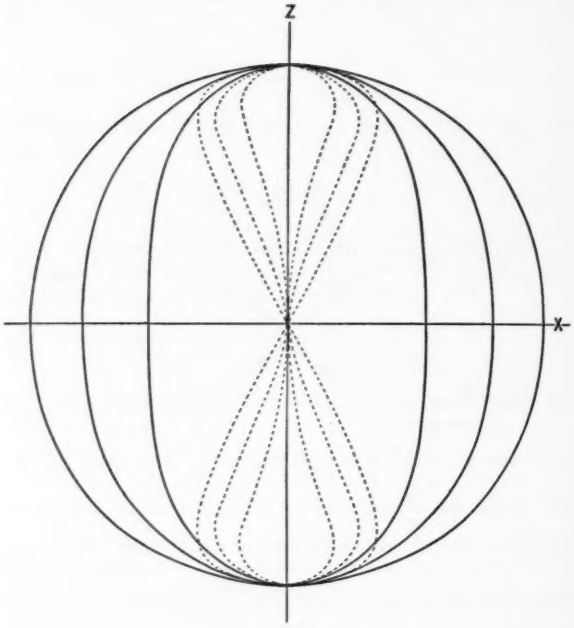

FIG. 2.

and the pinch points real and finite, the projection of the curves on the x, y and x, z planes are shown in Figs. 1 and 2. The dotted lines refer to left hand complexes.

For the quartic [2, 2] without a double generator [Cayley, I] the curves are of degree 8, and they pass through 8

* "Rappresentazione di una classe di superficie gobbe sopra un piano, e determinazione delle loro curve assintotiche." *Annali di matematica*, ser. 2, vol. 1 (1867–8).

pinch points, 4 on each double directrix. When the axes are rectangular, the cuspidal generators passing through the pinch points on the line z, w become asymptotes. When the pinch points are all real and distinct, a right hand curve and a left hand curve are shown in Fig. 3 and Fig. 4.

When the form [2, 2] has a double generator (Cayley, II) the asymptotic curves are of degree 6. There are now two pinch points on each line; this requires that, *e. g.*,

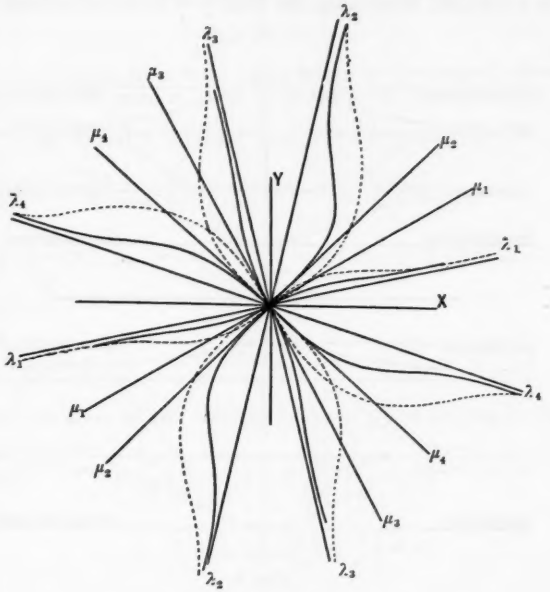

FIG. 3.

$\mu_2, \mu_3, \lambda_2, \lambda_3$ in Fig. 3, Fig. 4 all coincide. This form is unicursal. The form [3, 1] (Cayley, IX) is unicursal; it is of degree 6.

(β) *Directrices Coincident.*—The above method is not sufficient when the two directrices coincide. Suppose $n \leqq m$; from each point of the line issue n generators, all of which lie in a plane passing through the line. The points on the multiple line and the planes containing the line and the

generators issuing from the point are projective. Hence in this case also the surface belongs to a linear congruence, but the congruence is now special. It is necessary to have a pencil of complexes having the multiple line for directrix, and having the same homography as the given surface. Let the line x, y be the multiple directrix, then

$$xz' - x'z - p(yw' - y'w) = 0$$

is a complex containing the line; p is to be so determined

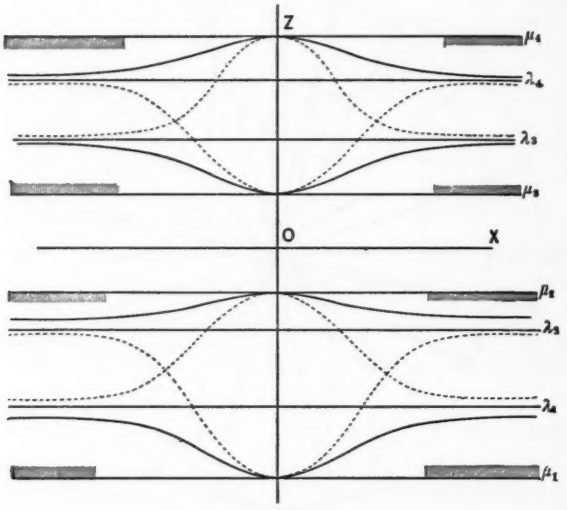

FIG. 4.

that as the point moves along the x, y axis, its polar plane will always coincide with the plane containing the corresponding generators of the surface. Finally $xy' - x'y = 0$ is a special complex with the multiple directrix for axis. By taking $p = 1$, the pencil of the complexes has the form

$$xz' - x'z - (yw' - y'w) + k(xy' - x'y) = 0$$

or, re-arranged,

$$(kx' + w')(yx' - y'x) + [x(y'w' - x'z') - x'(y'w - x'z)] = 0. \tag{7}$$

The equation of the surface now has the form

$$\varphi \equiv \sum_{r=0}^{n} v_2^r u_{m+n-2r} = 0$$

in which u_s is a binary quantic in x, y of degree s, and

$$v_2 \equiv yw - xz.$$

It is necessary to express the equation of the tangent plane to (8) at the point x', y', z', w' in the form

$$L(yx' - y'x) + M[x(y'w' - x'z') - x'(y'w - x'z)] = 0.$$

Since $$x' \frac{\partial \varphi}{\partial x'} + y' \frac{\partial \varphi}{\partial y'} = \sum_{r=0}^{n} v_2^r (m + n - 2r) u_{m+n-2r}$$

$$+ \sum_{r=0}^{n} r v_2^r u_{m+n-2r} = (m+n) \sum_{r=0}^{n} v_2^r u_{m+n-2r} - \sum_{r=0}^{n} r v_2^r u_{m+n-2r},$$

we have

$$x' \frac{\partial \varphi}{\partial x'} + y' \frac{\partial \varphi}{\partial y'} = - \sum_{r=0}^{n} r v_2^r u_{m+n-2r} = - \left(z' \frac{\partial \varphi}{\partial z'} + w' \frac{\partial \varphi}{\partial w'} \right).$$

The equation of the tangent plane can now be put in the form

$$y' \frac{\partial \varphi}{\partial y'} \left(\frac{y}{y'} - \frac{x}{x'} \right) + w' \frac{\partial \varphi}{\partial w'} \left(\frac{w}{w'} - \frac{z}{z'} \right)$$

$$+ \sum_{r=0}^{n} r v_2^r u_{m+n-2r} \left(\frac{z}{z'} - \frac{x}{x'} \right) = 0;$$

but $$\frac{(y'w' - x'z')}{y'} \frac{\partial \varphi}{\partial w'} = \sum_{r=0}^{n} r v_2^r u_{m+n-2r},$$

hence

$$y' \frac{\partial \varphi}{\partial y'} (yx' - y'x) - \frac{\partial \varphi}{\partial w'} [x(y'w' - x'z') - x'(y'w - x'z)] = 0. \tag{9}$$

The eliminant of (9) and (7) is, after a slight reduction,

$$\frac{\partial \varphi}{\partial y} + (kx + w)\frac{\partial \varphi}{\partial v_2} = 0 \tag{10}$$

which together with (8) defines the asymptotic lines.

Of the properties of these lines, II no longer applies, as the double lines coincide. The complexes are no longer coaxial, and the sign of the complex does not change with that of k; it is evident from equation (10) that every generator is cut but once (except at its intersection with the multiple line) by every asymptotic line. The property I, however, still holds, as may be proved as follows: All of the pinch points are now on the single multiple line, and there are $2m(n-1)$ of them. The direction of the plane $\lambda x = y$ is uniquely determined for each one of these points; the corresponding values of λ are determined from the vanishing of the discriminant of the binary equation

$$\varphi(x, \lambda w - z) = 0$$

or may be defined as those values which satisfy

$$\varphi = 0, \frac{\partial \varphi}{\partial v_2} = 0.$$

From equation (10), by replacing y by λx, it is seen that $\frac{\partial \varphi}{\partial v_2} = 0$ when $x = 0$, which proves that all the asymptotic lines pass through the pinch points, or it can be shown by eliminating z between (10) and (8); the result is the projection of the lines on the x, y, w plane. The coefficient of the highest power of w, equated to zero, will define the directions of the curves at the vertex $x = 0$, $y = 0$; these directions are independent of k and are easily shown to be the roots of the v_2 discriminant of φ.

To obtain the order of the curves in this case, the same method as before will suffice, when it is observed that the multiple line is itself a generator of multiplicity $m - n$. In (8) if λx be substituted for y, the directions of the n generators through the point $x = 0$, $z = \lambda w$ are given by the equation

$$u_{m+n}(\lambda) \cdot x^n + \cdots + (\lambda w - z)^n u_{m-n}(\lambda) = 0,$$

and one line will be $x = 0$ for every value of λ that satisfies $u_{m-n}(\lambda) = 0$; at these points one of the generators coincides with the multiple line. There are $2m(n-1)$ pinch points and these $m - n$ new points must count twice. For the

asymptotes there touch the multiple line; finally, every one of the n generators is cut by the same asymptotic line in but one point not on the directrix; the whole order is

$$2m(n-1)+2(m-n)+n-2\delta=2mn-n-2\delta.$$

In particular, if the surface is unicursal, the order reduces to $2m+n-2$.

For the surface $[2, 2]$ without double line (Cayley, IV) the order is 6, with a double line (Cayley, V) the order is 4. For these surfaces the multiple line cannot be a generator.

The surface $[3, 1]$ (Cayley, VI) is unicursal; its asymptotic lines are of order 5. The triple line is double generator.

For the cubic $[2, 1]$ (Cayley's cubic scroll) the order is 3; the double directrix counts as single generator.

CORNELL UNIVERSITY,
November, 1898.

WILLSON'S GRAPHICS.

Theoretical and Practical Graphics. By F. N. WILLSON, C.E., M.A., Professor of Graphics, Princeton University. New York, The Macmillan Company, 1898. 4to, 300 pp. Price, $4.00.

THIS work, primarily, is a text book on graphics compiled by an experienced teacher to meet the needs of his own classes. Few student have, heretofore, been called upon to make larger expenditures for books than has the embryo engineer and in combining, under the comprehensive title of graphics, much that is essential for such students, for example, chapters upon freehand and mechanical drawing, theory of the helix, link motion, trochoidal and other mechanical curves and the theory of descriptive geometry, a real need has been recognized. This volume is far more than a collection of class room notes. Every page bears evidence of conscientious care and research. The grouping of the chapters, the concise and useful table of contents, the clear cut and often elaborate illustrations and the exceptional typographical excellence cannot be too highly commended.

The book appeals at once to the artistic sense and to the admirers of descriptive methods, but, valuable as are these qualities, a review of them in a strictly mathematical journal would be out of place were it not for the fact that the treatment is mathematical, involving problems of interest to students of geometry.

In the opening paragraphs of Chapter I, much space is devoted to a consideration of the nomenclature, an effort being made to harmonize the "positional" properties discussed in descriptive geometry with those that would be defined by Cayley or Clebsch as projective. This will undoubtedly subject the author to criticism. Descriptive geometry is defined as "that branch of mathematics in which figures are represented and their *descriptive* properties investigated and demonstrated by means of projection." Descriptive properties are defined to be such as remain unaltered by projection, and, as one example, we find stated: "If a line is perpendicular to a plane, any plane containing the line will also be perpendicular to the plane." Concluding, the author says: "The main province of projection is obvious." Perpendicularity is not a projective property as defined by Cayley and Cremona; indeed, in all Euclid it is difficult to find one that is, Proposition 2, Book XI, being a conspicuous exception. Moreover, other methods than those of projection, the invariant theory, for example, are apt to be employed to demonstrate projective properties. The work in fact has little to do with such properties and the word *positional*, used by the author as synonymous, had better be substituted and otherwise defined if the definition of descriptive geometry as given by Church is to be replaced. This misconception is unfortunate, especially as, in general, the author has appreciated the importance of accurate definitions. When projections are made upon two perpendicular planes we have the orthographic projection first logically investigated by Gaspard Monge; and, reserving the general definition, the author calls this special branch to be considered "Monge's descriptive," an appropriate and historically valuable limitation.

It is evident that the author is at one time writing a treatise and at another a text book, and as a consequence the reviewer is sometimes at a loss to determine from what standpoint to criticise his work. For example, the quality of the paper, the elaborate and beautiful illustrations and the large and unusually fine quality of type would be out of place in a treatise, while desirable in a work intended to afford to draftsmen a daily illustration of neatness, precision. and the power of artistic expression.

Again, the numerous historical and explanatory notes accompanying the text are very desirable for the student, but at times are so worded as to leave the reader in doubt as to whether they have been verified or not. For example, we find on page 31, a construction for obtaining approximately a straight line equal in length to any semicircle and the accompanying note reads: "According to Böttcher it is due to a Polish Jesuit, Kochansky, and was first published in the Acta Eruditorum Lipsiae, 1685." Again in article 114, the sections of an "annular torus" made by a plane parallel to but not containing the axis are said to cut the surface in "Cassian ovals." Why Cassian instead of Cassinian, which last is the generally accepted spelling, and is the author quite sure of the truth of his statement?

The equation of the torus (Salmon, Geometry of Three Dimensions, 2d edition, p. 348) is

$$(1) \qquad (x^2+y^2+z^2+a^2-r^2)^2=4a^2(x^2+y^2).$$

If $y=c$, we may write

$$(2) \qquad (x^2+z^2+c^2+a^2-r^2-2ax)(x^2+z^2+c^2+a^2-r^2+2ax)=4a^2c^2,$$

or

$$ss'=\text{constant},$$

s and s' denoting circles. The sections, it is true, are declared, in the article quoted above, to be "*lemniscates of various kinds.*" The ovals of Cassini might, therefore, by inference be included. But, in the judgment of the reviewer, Willson has been misled by Salmon, who errs in this statement, for the accepted equation of the Cassinian is

$$(3) \qquad (x^2+z^2+a^2)^2-4a^2x^2=m^4$$

(Salmon, Higher Plane Curves, p. 44, ex. 3).

Equation (2) has three constants and (3) has but two and if these definitions are to hold we must have $y=r$, giving (3), which is then a Cassinian for every value of a, reducing to the lemniscate of Bernouilli only when $a=m$.

In Chapter V, keeping always in view the double object of the book, namely, to afford examples for the draftsman and to develop mathematical principles of importance to the engineer, the author discusses in turn the helix, the conic sections, link motion curves, centroids, trochoids, limaçon, cardioid, trisectrix, spirals, conchoid, quadratrix, witch,

Cartesian and Cassinian ovals and the catenary. Well known as these curves have become, Professor Willson with new and beautiful figures, interesting descriptions, and notes has made out of this material a delightful chapter that a mathematician must read with pleasure. The trochoids are treated with unusual elaboration. Here, again, the author is writing a treatise, for, after developing the subject extensively for ten pages, an appendix, primarily devoted to a reprint of his classification of the trochoids, is added to a volume already too large to be easily handled by the student.

One hundred pages are devoted to a discussion of descriptive geometry and its applications. Compared with the treatise of Mannheim it might suffer, but as an English text-book it is a decided improvement on its predecessors. Only a glance is necessary to convince the student familiar with Church, for example, that most, if not all, of the old difficulties and special cases are here explained with clearness. Indeed the one criticism may be that too little is left for the student's ingenuity and for the cultivation of his imagination, the faculty deriving the most benefit from this study.

As interesting examples of the surfaces considered the following are noted: Of the third order, the conoid of Plücker; of the fourth, the cyclide of Dupin, the conocuneus of Wallis, the cylindroid of Frézies, the corne de vache, the conchoidal hyperbola of Catalan and the torus; in addition, many that are transcendental.

Chapter X is devoted to practical applications of the principles developed, treating in turn of projections and intersections by the third angle method, the development of of surfaces, projections, intersections, and tangencies of developable, warped, and double curved surfaces.

It is by far the most interesting chapter and contains all that is needed in this connection by the undergraduate student of engineering. The treatment of perspective is elaborate but the absence, in a work otherwise well supplied in this respect, of notes connecting the subject with covariants is worthy of comment. We have indeed an illustration of the fact that, whereas the pure mathematician has but slight knowledge of the draftsman's methods, the engineer on the other hand is rarely well informed concerning the algebra of his subject.

The five pages of Chapter XI are devoted to the graphical solution of spherical triangles, constituting a departure from usual methods that is theoretically interesting but likely to be omitted by most teachers using the book.

Orthographic, stereographic, and allied projections of the spherical surface form the subjects of Chapter XII, and the next three deal with the principles of shade, shadow, and perspective.

While it is not written expressly for mathematicians, no student of geometry, especially of surfaces, can review this book without profit. In the main it is firmly grounded on mathematical principles accompanied with illustrations of the best practice of modern draftsmen.

J. B. CHITTENDEN.

COLUMBIA UNIVERSITY.

PASCAL'S REPERTORIUM OF HIGHER MATHEMATICS.

Repertorio di matematiche superiori. I. Analisi. Per ERNESTO PASCAL, Professor ordinario nella R. Università di Pavia. Milan, U. Hoepli, 1898. 16mo, xv + 643 pp.

THAT the increasing need of encyclopædic literature in mathematics is being promptly met is indicated by the appearance during the last decade of Carr's synopsis, Láska's compendium, the first two volumes of Hagen's synopsis, and most recently the first volume of Pascal's repertorium and the first part of Burkhardt and Meyer's encyclopædia. The body of doctrine of mathematics is expanding at such a rapid rate that the difficulty of securing a liberal education in the science is becoming wellnigh insurmountable, a difficulty that is obvious on contemplating the marvelous development centering about the notions function and group, to cite particular examples. In no subject is special specialization growing more imperative than in mathematics; in the midst of difficulty and demand the student should hail with delight the valuable services of a work so admirably adapted to purposes of orientation as Professor Pascal's repertorium promises to be.

The author's plan, adhered to without deviation, is to present with regard to each theory of modern mathematics the fundamental definitions and notions, the characteristic necessary theorems and formulæ, and citations to the principal works of its bibliography. The definitions are clear and unequivocal; the statements of the theorems, always given

without demonstration, are concise and unambiguous; and the bibliographical references* are sufficiently full to supply the needs of the general mathematical reader.

The subject matter of the first volume, devoted to analysis as announced in the title, is arranged in chapters in the following order: 1° introductory theory of higher algebra, thirty-six pages; 2° theory of substitution groups, sixteen pages; 3° theory of determinants, twenty pages; 4° theory of series, infinite products, and continued fractions,* twenty-six pages; 5° theory of algebraic equations, thirty-two pages; 6° differential calculus, twenty-six pages; 7° integral calculus, thirty-four pages; 8° differential equations, thirty-eight pages; 9° theory of transformation groups, thirteen pages; 10° calculus of finite differences, fifteen pages; 11° calculus of variations, eighteen pages; 12° theory of invariants, forty-three pages; 13° functions of complex variables, thirty pages; 14° the theory of functions in relation to the theory of groups; periodicity and automorphism, twenty-six pages; 15° algebraic functions and abelian integrals, thirty pages; 16° theory of elliptic functions, fifty-four pages; 17° hyperelliptic and abelian functions, nineteen pages; 18° special functions, fifty-eight pages; 19° analytical representation of functions, fourteen pages; 20° theory of integral numbers, rational or complex, forty-three pages; 21° theory of algebraic and transcendental numbers, thirteen pages; 22° calculus of probabilities, thirty-two pages; 23° analytical instruments and apparatus, sixteen pages.

There is space to specify the contents of no more than a few of the chapters. Those commented upon have been selected either because their subjects, as stated in the paragraph above, give but a vague idea of their material or because their presence calls for special notice. The introductory theories of the first chapter comprise those of irrational numbers, complex numbers, quaternions, groups of points, aggregates and ensembles, general concept function, rational

*The extent of the bibliographical notes may be seen by observing that, in the chapter on differential equations, page 192, the existence theorem is accompanied by references relative to its demonstration to Moigno's differential calculus, vol. 2 (1844), for Cauchy's proof; to Briot and Bouquet, *Jour. de l'Éc. Poly.*, vol. 36; to Lipschitz, *Ann. di mat.*, vol. 2, *Darboux's Bulletin*, vol. 10; to Volterra, *Battaglini's Journal*, vol. 19; to Peano, *Acc. Torino* (1886), *Math. Annalen*, vol. 37; to Arzelà, *Acc. Bologna* (1896).

*To the list of corrections given on page 643 should be added that in the head-lines of pages 93 and 94 "funzione" should be replaced by "frazione."

integral functions of one variable, limits, superior and inferior limits of the values of a function, continuous and discontinuous functions, combinations, and binomial coefficients.

In passing we remark as noteworthy features the sections on functions appertaining to substitution groups, and on the analytical representation of substitutions in the second chapter; the collection of special determinants including Pfaffians, Wronskians, Jacobians, and Hessians of the third chapter; the convergence criteria and particular series of the fourth chapter; the résumé of Galois's theory in the chapter on the theory of equations; the discussion of Rolle's theorem and of the Taylor-Maclaurin formula, together with the tables of integrals, in the chapters on the infinitesimal calculus. The chapter on differential equations is well written, but the difficulties of the theory of partial differential equations have severely taxed the author's powers of exposition and condensation.

The recent numbers of the Italian journals give evidence of the favorable reception and productivity of Lie's theories at the hands of the Italian mathematicians, and it is equally refreshing to find a chapter of this book devoted to Lie's work, even though the chapter be compressed to thirteen pages. The author considers in turn groups of point transformations, finite and differential invariants of groups of one parameter,* contact transformations, and differential invariants and parameters. Relative to the theory of differential invariants and parameters both categories of research are presented, namely those which spring directly from Lie's theory of groups, and those whose origin is in the theory of differential forms. The bibliographical references include Lie's latest work on the geometry of contact transformations.

The interpolating function and approximate quadratures are adequately treated in the chapter on finite differences; among the interesting details of the eleventh chapter is a a collection of the historic problems of the calculus of variations. The chapter on invariants, one of the most complete in the book, recapitulates the theory in seven sections: binary forms, symbolic representation; invariants and covariants; Clebsch-Gordan formula; vocabulary of the various terms employed in the theory of forms; complete systems of invariant forms; typical representations of binary forms, associated forms; canonical representation of forms.

* The term "r-gliedrige" of Lie's terminology, which usually appears in English and French as "r parameter" or "of r parameters," Pascal translates "di classe r."

The thirteenth chapter presents the elements of the theory of functions of a complex variable in a most logical manner and the discussion of the definition of an analytical function is one of the most useful digressions in the book. It was a happy thought of the author's that led him to include a chapter on the theory of functions related to the theory of groups in which he has incorporated linear substitutions, groups of linear substitutions, anharmonic groups, polyedral functions and groups, periodic functions, modular functions, and the functions of Fuchs and Klein. The chapter on algebraic functions and abelian integrals, after generalities on algebraic functions and Riemann surfaces, presents the properties of abelian functions of the first, second, and third species and concludes with Abel's theorem. The ϑ function of Jacobi, elliptic functions of Legendre, the σ and p functions of Weierstrass, and rational functions of p and p' constitute the material of the chapter on elliptic functions which terminates with the theory of the transformation and the multiplication of these functions. In addition to the theorem of inversion of Jacobi and the general properties of abelian functions, the chapter on hyperelliptic functions has sections on the ϑ functions and their properties, the ϑ functions whose arguments are abelian functions of the first species, and the σ of Klein in the hyperelliptic case. Under the heading special functions the author has constructed the largest chapter of the book; it includes the discussion of the exponential, logarithmic, circular, hyperbolic, and hypergeometric functions and the functions of Euler, Bernouilli, Legendre, Laplace, Bessel, and Lamé. The series of Wronski, of Lagrange, and of Fourier, together with developments of functions in series of functions of Legendre, Laplace, and Bessel, make up the chapter on the analytical representation of functions which is the last chapter devoted to the theory of functions.

The twentieth and twenty-first chapters have to do with the theory of numbers. The author's arrangement is as follows: Divisibility of rational integral numbers, prime numbers, the numerical function $E(x)$, generalities upon congruences, congruences of the first degree, congruences of the second degree, quadratic residues, binominal congruences, residues of the third and higher orders, exponential congruences, primitive roots and indices, quadratic forms, representation of numbers by means of forms, integral complex numbers of Gauss, integral cubical complex numbers, generalities on algebraic numbers, divisibility of algebraic numbers, ideal numbers of Kummer, transcendental numbers,

the number π (the number e is treated in connection with the exponential function in the eighteenth chapter).

The usual elements of the theory of probabilities and its relation to the theory of errors are the principal subjects of the twenty-second chapter. The last chapter consists of three sections occupied respectively with the mechanical instruments of arithmetic, of algebra, and of the integral calculus; among the last are Lord Kelvin's harmonic analyzer and the integraph of Abdank-Abacanowicz.

The author undoubtedly has had many perplexing problems to settle relative to the admission or rejection of numerous details that suggested themselves, but the reader is occasionally at a loss to explain omissions among which may be mentioned for example that of Horner's method of approximation of the real roots of an algebraic equation, that of examples of derivativeless functions, and that of the theory of integral invariants which might have been introduced in its proper place as a corollary of Lie's theory of differential invariants or presented under Poincaré's treatment. The absence of an analytical index of notions and names is perhaps to be accounted for by limits previously set on the size of the book; it is to be regretted that the publisher did not use for this purpose the part taken up by a catalogue the majority of whose entries possess only a very secondary interest to the student of mathematics.

Relative to this modest volume it is observed finally that it is an unusual piece of work as a product of the bookmaker's art. To be sure the margins are narrow, but the type is large and clear, the pages are not wide enough to tire the eye, and, although there are more than seven hundred pages (a catalogue of the Hoepli manuals is appended) in substantial cloth binding, the perusal of the book is accompanied by no more muscular fatigue than attends that of an ordinary paper bound volume of half the number of pages. These mechanical points will contribute in no small degree to the usefulness of the book and they are worthy of consideration by those who are contemplating the making of mathematical text books.

The companion volume on geometry will be awaited with impatience by all who have availed themselves of the volume on analysis. Its early appearance is promised in a preliminary announcement which proposes the following list of chapters: 1° geometry of the fundamental forms; 2° conics; 3° quadrics; 4° general theory of curves and surfaces; 5° plane cubics; 6° plane quartics; 7° space cubics; 8° space quartics; 9° surfaces of the third order; 10° sur-

faces of the fourth order ; 11° geometry of the straight line ; 12° ruled surfaces, complexes, and congruences ; 13° differential geometry ; 14° non-euclidean geometry ; 15° geometry of space of n dimensions ; 16° kinematical geometry ; 17° theory of connexes.

EDGAR ODELL LOVETT.

PRINCETON, NEW JERSEY.

D'OCAGNE'S DESCRIPTIVE AND INFINITESIMAL GEOMETRY.

Cours de Géométrie descriptive et de Géométrie infinitésimale. Par MAURICE D'OCAGNE, Ingénieur des Ponts et Chaussées, Professeur à l'École des Ponts et Chaussées, Répétiteur à l'École Polytechnique. Paris, Gauthier-Villars et Fils, 1896. 8vo, xi+428 pp.

THIS work is an expansion of the course in pure geometry given by the author at the École des Ponts et Chaussées. It proposes to give an exposition of all the geometrical notions which are of interest to engineers, the material not required in the course appearing in small print. As the title indicates, the construction of the book is unique. The author deems it necessary to separate completely that which has to do with the representation of geometrical bodies from that which treats of their intrinsic properties. Accordingly his course is divided into two parts represented in the work by the two distinct divisions : *Géométrie descriptive* and *Géométrie infinitésimale*. The author believes also that the exposition of general doctrines should precede that of the details of a subject and prefaces each chapter with a body of essential principles before examining any particular case ; thus for example, in the theory of surfaces he presents an ensemble of general properties before studying surfaces of a special nature, such as the *surfaces gauches* ; this is contrary to the custom prevailing in similar courses.

1. The first part (two hundred and forty-six pages) of the work includes the first four chapters. In the first chapter, *Projections cotées*, we have the usual details relative to the representation of the ordinary relations between right lines and planes, together with certain problems concerning the round bodies, and the theory of topographical surfaces and profiles.

The second chapter, *Perspective axonométrique*, develops a special mode of plane representation of the bodies of space with a view to the application to be made of it in the theory of shadows. The theory is developed more at length than is necessary for the purposes of the theory of shadows. The axonometric perspective of the plane and that of ordinary space are naturally treated separately. The theory of d'Ocagne encounters a little difficulty in giving a rigorous definition of the apparent contour of a body. The chapter is supplemented by two notes one of which shows the identity of axonometric perspective and oblique projection, and the other indicates the application of this mode of perspective to the representation of the *motifs* of architecture.

The third chapter is occupied with the *Théorie des ombres usuelles*. We remark the careful coördination of the general definitions, methods, and theorems systematically arranged in the first section of the chapter; the very simple rule in the second section to which the author refers the determination of the shaded parts of a polyhedron; and the general method of the third section derived from the theory of the preceding chapter for the construction of shadows thrown upon planes. The fourth section applies the method of oblique projection to the study of shadows on surfaces, studying in order cylinders, cones, spheres, and surfaces of revolution. The chapter is amply illustrated, as in fact is the entire volume: three hundred and forty being the number of figures distributed through the text, some of which are duplicated.

The last chapter of the first part is devoted to the ordinary theory of linear perspective. The initial explanation of the object and character of geometrical perspective puts the student on his guard against current misconceptions of perspective. The author's perspective of the sphere and the application of his transformation* to the construction of conical perspective, are worthy of special notice.

2. The second division of the book is prefaced by a preamble whose purpose as stated by the author in his introduction to the volume is twofold: 1° to give certain indispensable explanations relative to the infinitesimal elements in the sequel, in such a manner as to attribute the same rigor to infinitesimal formulæ, *i. e.*, formulæ containing infinitesimals, obtained geometrically, as to those to which we are led by analysis; 2° to define that which is meant by the geometry of variable forms, independently of every idea of displacement. With regard to this second point the

*See d'Ocagne, *Nouv. ann. de math.*, 3d series, vol. 12 (1893), p. 350.

author maintains by way of illustration that to say that a circle is the locus of points equidistant from a given point or that a circle is a curve described by a point which remains at a constant distance from a given point, is to make one and the same statement. In the one case we figure simultaneously, in the other successively, all the points of the locus.

The fifth chapter, *Courbes planes*, takes as its basis certain fundamental formulæ borrowed from Mannheim,* but stated in new form and demonstrated in novel manner conformably to the point of view taken in the preface. With the exception of the one relative to the centers of curvature of conics, the numerous applications given in the second section are taken from Professor d'Ocagne's original papers: for example, 1° a theorem on the determination of normals,† generalizing the known theorem on the construction of normals to curves and surfaces defined by a relation between the distances of their points to fixed curves and surfaces; 2° the study‡ of the envelope of a chord subtending in a curve an arc of constant length and of that of a chord viewed from a fixed point under a constant angle; 3° the author's idea of associating with a given curve another curve so related to the former that the infinitesimal elements of the first are derivable from infinitesimal elements of a lower order of the second, for example, normals of the first from points of the second, and so on; for a given curve there exists a great number of curves possessing this character;§ the one seeming to possess the greatest advantage in the way of applications is d'Ocagne's adjoined|| curve of normal directions.

The chapter closes with an application to kinematics¶ leading to the theory of Peaucellier's cell and of the articu-

* Mannheim, Cours de géométrie descriptive de l'École Polytechnique, 2d edition, pp. 203-205.

† d'Ocagne, *Comptes rendus*, 2d semester, 1889, p. 959; *Nouv. ann. de math.*, 1890, p. 289; 1894, p. 501.

‡ *Nouv. ann. de math.*, 1883, p. 252; 1886, p. 88.

§ d'Ocagne, "Remarques sur la géométrie infinitésimale des courbes planes," *Jour. de math. spéciales*, 1888.

|| In the plane of the curve (M) take two poles O and D; draw the radius vector OM and through the pole D, called the pole of normal directions, a parallel DM' to the normal MN; the locus (M') of the point M' is the adjoined curve of normal directions of the curve (M). See *Amer. Jour. of Math.*, 1888, p. 55; 1892, p. 227; *Bulletin de la Soc. Math. de France*, 1892, p. 49.

¶ d'Ocagne, *Nouv. ann. de math.*, 1881, p. 456; 1882, p. 40; 1884, p. 199.

lated systems† of Hart and Kempe. The author remarks in the introduction that this last application shows how the science of kinematics can be attached to the geometry of variable figures when the latter is conceived without the intervention of the notion of displacement, a conception, in in the author's opinion, more satisfactory from a philosophical point of view.

The sixth chapter studies the theory of *Courbes gauches.* It consists of two sections, one on the general principles and one on the helix. The elements of a space curve, osculating plane, curvature, torsion, osculating sphere, osculating helix, and so on, are presented and applied completely to the general helix. There is a very elegant demonstration given of the theorem that the oblique projection of a circular helix on a plane perpendicular to its axis is an cycloid.

The seventh chapter deals with *Surfaces en général.* The author very properly calls attention to his elementary demonstration of the theorem of Malus in the first section. In studying the curvature of lines traced on a surface in the second section, considering first the sense of the curvature, the indicatrix of Dupin is introduced to determine the changes of sense; considering the magnitude of the curvature, the author establishes first the formula for any curve of double curvature, using the same order of procedure as Jordan adopts in his Cours d'Analyse, and then refers the question, by the theorems of Meusnier and Euler, to the determination of the principal radii of curvature, finally employing Dupin's indicatrix to determine the variations in the magnitude of the curvature. After introducing the notions, axes of curvature and deviation, together with the theorem of Sturm and the formulæ of Bertrand and Bonnet, the author presents an interesting paragraph on the curvature of surfaces including Casorati's recent definition* along with those of Gauss and Sophie Germain. The third and last section attacks lines traced on a surface by defining geodesic curvature and torsion; the fundamental formulæ relative to the latter element are deduced immediately from those relative to deviation derived in the preceding section. By a very elementary process the geometric theory of lines of curvature presents itself and the author establishes Dupin's theorem on triply orthogonal systems in a simple and

* Liguine, Ibid., 1882, p. 153.

† *Acta Math.*, vol. 14, p. 95. Casorati's form is equal to one half the sum of the squares of the reciprocals of the principal radii of curvature; for this reason d'Ocagne proposes that it be called the mean quadratic curvature.

elegant manner. The property of the osculating plane's tangency to the surface is adopted as the defining property of asymptotic lines; their curvature is determined by means of a theorem of Beltrami, the theorem of Meusnier failing for these lines; their torsion is found by a theorem of Enneper. Geodesic lines are defined as those whose osculating plane is normal to the surface; their essential property of minimum length and their analogies with straight lines of the plane are fully indicated. The author suggests that those curves whose points are at a constant geodesic distance from a fixed point be called geodesic circumferences. This change of terminology is worthy of remark; the term geodesic circle has been ambiguously used to designate the curves of constant geodesic distance studied by Gauss and those of constant geodesic curvature first considered by Minding. After an account of curvilinear coördinates and an historical résumé of the details of the theory of minimal surfaces the chapter concludes with a brief summary of the theory of applicability.

The eighth and last chapter of the work is entitled *Surfaces de nature spéciale*. In coördinate geometry surfaces are classified according to the nature of their equations; similarly in infinitesimal geometry surfaces are arranged into certain families with reference to their infinitesimal properties. Of these families the envelopes of spheres and in particular surfaces of revolution are taken up in the first section. The treatment of skew surfaces in the second section is characterized by a careful consideration of the parameter of distribution which results in full rigor, generality, and precision in the delineations of the problems treated. The author employs his method of orthogonal tangents to construct tangent planes to skew surfaces having a plane director. The study of skew surfaces having as director a cone of revolution appears in a new form. Another original stroke in this chapter is the complete determination, by linear constructions, of the indicatrix of every point of a skew helicoid having a cylindrical core. The third section concludes the chapter with developable surfaces treated as a particular variety of ruled surfaces.

EDGAR ODELL LOVETT.

PRINCETON, NEW JERSEY.

SOPHUS LIE.*

BY PROFESSOR GASTON DARBOUX.

SOPHUS LIE was born on the 17th of December, 1842, at Nordfjordeid (near Florö) where his father, John Herman Lie, was pastor. The studies of his childhood and youth did not reveal in him that exceptional aptitude for mathematics which is signalized so early in the lives of the great geometers: Gauss, Abel, and many others. Even on leaving the University of Christiania in 1865, he still hesitated between philology and mathematics. It was the works of Plücker on modern geometry which first made him fully conscious of his mathematical abilities and awakened within him an ardent desire to consecrate himself to mathematical research. Surmounting all difficulties and working with indomitable energy he published his first work in 1869, and we can say that from 1870 on he was in possession of the ideas which were to direct his whole career.

At this time I frequently had the pleasure of meeting and conversing with him in Paris where he had come with his friend F. Klein. A course of lectures by Sylow revealed to Lie all the importance of the theory of substitution groups; the two friends studied this theory in the great treatise of our colleague Jordan; they saw fully the essential rôle which it would be called upon to play in all the branches of mathematics to which it had then not been applied. They have both had the good fortune to contribute by their works to impressing upon mathematical studies the direction which appeared to them to be the best.

A short note of Lie "Sur une transformation géométrique," presented to our Academy in October 1870 contains an extremely original discovery. Nothing resembles a sphere less than a straight line and yet, by using the ideas of Plücker, Lie found a singular transformation which makes a sphere correspond to a straight line, and which consequently makes possible the derivation of a theorem relative to an ensemble of spheres from every theorem relative to an aggregate of straight lines, and vice versa. It is true that if the lines are real, the corresponding spheres are imaginary. But such difficulties are not sufficient to deter geometers. In this curious method of transformation, each property relative to asymptotic lines of a surface is transformed into a property relative to lines of curvature. The name of Lie will remain attached to these concealed relations which

* Translated from the *Comptes rendus*, February 27, 1899, by Professor EDGAR ODELL LOVETT.

connect the two essential and fundamental elements of geometric investigation, the straight line and sphere. He has developed them in detail in a memoir full of new ideas which appeared in 1872 in the *Mathematische Annalen*.

The works following this brilliant beginning fully confirmed all the hopes to which it gave birth. Since the year 1872 Lie has put forth a series of memoirs upon the most difficult and most advanced parts of the integral calculus. He commences by a profound study of the works of Jacobi on the partial differential equations of the first order and at first coöperates with Mayer in perfecting this theory in an essential point. Then, by continuing the study of this beautiful subject, he is led to construct progressively that masterful theory of continuous transformation groups which constitutes his most important work and in which, at least at the start, he was aided by no one. The detailed analysis of this vast theory would require too much space here. It is proper, however, to point out particularly two elements wholly essential to these researches: first, the use of contact transformations which throws such a vivid and unexpected light upon the most difficult and obscure parts of the theories relative to the integration of partial differential equations; second, the use of infinitesimal transformations. The introduction of these transformations is due entirely to Lie; their use, like that of Lagrange's variation, naturally greatly extends both the notion of differential and the applications of the infinitesimal calculus.

The construction of so extended a theory did not satisfy Lie's activity. In order to show its importance he has applied it to a great number of particular subjects, and each time he has had the good fortune of meeting with new and elegant properties. I find my preference in the researches which he has published since 1876 on minimal surfaces. The theory of these surfaces, the most attractive perhaps that presents itself in geometry, still awaits, and may await a long time, the complete solution of the first problem to be proposed in it, namely, the determination of a minimal surface passing through a given contour. But, in return, it has been enriched by a great number of interesting propositions due to a multitude of geometers. In 1866 Weierstrass made known a very precise and simple system of formulæ which has called forth a whole series of new studies on these surfaces. In his works Lie returns simply to the formulæ of Monge; he gives their geometric interpretation and shows how their use can lead to the most satisfactory theory of minimal surfaces. He makes known methods which permit

of determining all algebraic minimal surfaces of given class and order. Finally, he studies the following problem : to determine all algebraic minimal surfaces inscribed in a given algebraical developable surface. He gives the complete solution for the case where only one of these surfaces inscribed in the developable is known.

Of great interest also are the researches which we owe to him on the surfaces of constant curvature, in the study of which he makes use of a theorem of Bianchi on geodesic lines and circles, likewise those on surfaces of translation, on the surfaces of Weingarten, on the equations of the second order having two independent variables, et cetera. I should reproach myself for forgetting, even in so rapid a résumé, the applications which Lie has made of his theory of groups to the non-euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge.

These extensive works quickly attracted to the great geometer the attention of all those who cultivate science or are interested in its progress. In 1877 a new chair of mathematics was created for him at the University of Christiania, and the foundation of a Norwegian review enabled him to pursue his work and publish it in full. In 1886, he accepted the honor of a call to the University of Leipzig ; he taught in this university with the rank of ordinary professor from 1886 to 1898. To this period of his life is to be referred the publication of his didactic works, in which he has coördinated all his researches. Six months ago he returned to his native land to assume at Christiania the chair which had been especially reserved for him by the Norwegian parliament, with the exceptional salary of ten thousand crowns. Unfortunately, excess of work had exhausted his strength and he died of cerebral anæmia at the age of fifty-six years.

Nowhere is his loss felt more keenly than in our country, where he had so many friends. True, in 1870 a misadventure befell him, whose consequences I was instrumental in averting. Surprised at Paris by the declaration of war, he took refuge at Fontainebleau. Occupied incessantly by the ideas fermenting in his brain, he would go every day into the forest, loitering in places most remote from the beaten path, taking notes and drawing figures. It took little at this time to awaken suspicion. Arrested and imprisoned at Fontainebleau, under conditions otherwise very comfortable, he called for the aid of Chasles, Bertrand, and others ; I made the trip to Fontainebleau and had no trouble in con-

vincing the procureur impérial ; all the notes which had been seized and in which figured complexes, orthogonal systems, and names of geometers, bore in no way upon the national defenses. Lie was released ; his high and generous spirit bore no grudge against our country. Not only did he return voluntarily to visit it but he received with great kindness French students, scholars of our École Normale who would go to Leipzig to follow his lectures. It is to the École Normale that he dedicated his great work on the theory of transformation groups. A number of our theses at the Sorbonne have been inspired by his teaching and dedicated to him.

The admirable works of Sophus Lie enjoy the distinction, to-day quite rare, of commanding the common admiration of geometers as well as analysts. He has discovered fundamental propositions which will preserve his name from oblivion, he has created methods and theories which, for a long time to come, will exercise their fruitful influence on the development of mathematics. The land where he was born and which has known how to honor him can place with pride the name of Lie beside that of Abel, of whom he was a worthy rival and whose approaching centenary he would have been so happy in celebrating.

NOTES.

THE sixth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the State University of Ohio, Columbus, Ohio, on Friday and Saturday, August 25–26. The meeting will thus immediately follow that of the American Association for the Advancement of Science. A circular giving more definite information will shortly be issued by the committee in charge.

A NEW List of Members of the AMERICAN MATHEMATICAL SOCIETY, including the constitution, by-laws, and the reports of the Treasurer and Librarian, has recently been published and distributed to the members of the Society. Copies of the List may be obtained from the Secretary.

AT the regular meeting of the London mathematical society of February 9th the following papers were presented : —"Note on a case of divisibility of a function of two vari-

ables by another function," by Mr. A. BERRY; "Scattering of electric waves by an insulating sphere," by Professor A. E. H. LOVE; "On groups of order p^3q," by Mr. A. E. WESTERN; "The group of linear substitutions on mq variables which is defined by a certain invariant," by Dr. L. E. DICKSON; "On the irreducible concomitants of any number of binary quartics," by Mr. A. YOUNG; "On a certain minimal surface and on a solution of $\nabla^2 U = 0$," by Mr. T. J. BROMWICH; "On the complete system of covariants of a single Pfaffian expression, and of a set of Pfaffian expressions," by Mr. J. BRILL; "On the figure of Jacobi with respect to a linear system of hyperquadrics," by Professor P. H. SCHOUTE.

A MEETING of the Edinburgh mathematical society was held February 10th, at which papers were read by Dr. SPRAGUE "On the eight queens' problem,' and by Professor J. E. A. STEGGALL "On a problem of Lewis Carroll's."

THE Annuario for 1898 of the Circolo matematico di Palermo shows that society to be in a flourishing condition with one hundred and seventy-four members of whom thirty-one are not residents of Italy. Professor F. CALDERARA is president, and Professors F. GERBALDI and G. B. GUCCIA are secretaries for the years 1898–99. Professor Guccia represents the council of the society as directing editor of the *Rendiconti*, of which the current volume is the thirteenth.

DR. GUSTAF ENESTRÖM, editor of the *Bibliotheca mathematica* has issued an elaborate index of the first ten volumes (1887–96) of the second series of that historical journal. The index occupies eighty-five octavo pages and is subdivided into four parts: an alphabetical list of authors, a systematic table of the original notes, a list of the reviews, and an alphabetical name and subject index. A unique feature of the index is the short biographical notes placed under the names in the list of authors; forty-three of these notices are accompanied by small well-executed photo-engravings of the authors.

THE concluding number has just been received of the first volume of the *Bolletino di bibliografia e historia delle scienze matematiche*, a quarterly review edited by Professor GINO LORIA, of the University of Genoa, and published by Carlo Clausen, of Turin. Each number is to consist of at least thirty-two octavo pages and the subscription price for the annual volume of four numbers is seven and a half francs for countries in the postal union. The first volume consists

of one hundred and sixty pages accompanied by a table of contents and an index of names; it contains two articles, one by the editor on the history of the strophoid, and the other an appreciative article in French on the scientific work and life of Tchébychef by A. VASSILIEF. In addition to these the volume contains a large number of reviews of recent mathematical tracts and treatises, together with announcements of university programmes, necrological notices, and general notes.

THE subject of the Adams prize of the University of Cambridge for 1901 is "Electric waves." The competition is open to all graduates of the university and the prize is two hundred and twenty-five pounds sterling.

THE prize problem of the Belgian academy of sciences for the current year calls for an important contribution to the geometry of the straight line without specifying what the general nature of the contribution is to be. The prize is six hundred francs. Memoirs should be written in French or Dutch and presented anonymously to the secretary of the academy before August 1st, 1899.

THE Madrid academy of sciences offers three prizes for a treatise on circular, hyperbolic, spherical, and spheroidal trigonometry. The first prize consists of fifteen hundred pesetas and a gold medal; the second, of a medal; and the third, of an honorable mention. Competing works should be written in Spanish or Latin and presented anonymously before December 31st, 1899.

THE Spring announcements of the Cambridge University press include the second volumes of the collected mathematical papers of Professor P. G. TAIT and of the scientific papers of the late Professor J. C. ADAMS; scientific papers by Lord RAYLEIGH, the late Dr. HOPKINSON, and Professor O. REYNOLDS; "A treatise on geometrical optics," by Mr. R. A. HERMAN; "On the kinetic theory of gases," by Mr. S. H. BURBURY.

UNIVERSITY OF PARIS. The following courses in mathematical subjects are announced by the faculty of sciences for the second semester beginning March 1st, 1899, each course consisting of two lectures per week:—By Professor PICARD: Differential equations from the point of view of mathematical physics.—By Professor GOURSAT: Analytical functions and differential equations.—By Professor APPELL: The general laws of the motion of systems, analytical me-

chanics, hydrostatics, and hydrodynamics.—By Professor POINCARÉ: The figures of the heavenly bodies and their motion about their centers of gravity.—By Professor WOLF: The subjects comprised in the programme for the certificate in astronomy.—By Professor BOUSSINESQ: The propagation of motion in an elastic medium of infinite dimensions.—By Professor G. KOENIGS: The study of machines.—By Professor LIPPMANN: Acoustics and optics.—By M. RAFFY: Differential equations and their applications to mechanics and physics.—Conferences will be conducted by Messrs. RAFFY, HADAMARD, PUISEUX, ANDOYER, and BLUTEL.

PROFESSOR G. FLOQUET has been made professor of mathematical analysis, and Dr. J. MOLK, professor of mechanics, at the University of Nancy. Dr. W. NATANSON, of Vienna, has been appointed assistant professor of mathematics in the University of Cracow.

DR. OTTO WIENER, of Giessen, has been called to a full professorship of physics at the University of Leipzig, and Dr. OTTO WIEDEBURG has been promoted to an assistant professorship of physics in the same institution. Professor W. WIEN, of Aachen, has been called to Giessen as Professor Wiener's successor. Professor WALTER KÖNIG, of Frankfort, has been made professor of theoretical physics at the University of Heidelberg.

PROFESSOR F. SCHILLING, of Karlsruhe, has accepted a call to Göttingen, to succeed Professor A. Schoenfliess who has gone to Königsberg.

THE deaths are announced of Professors W. HANKEL and G. WIEDEMANN of the University of Leipzig and of Dr. RUPERT BÖCK, professor of mechanics in the technical high school of Vienna.

AT Cambridge University, Mr. G. W. WALKER, of Trinity College, has been elected to the Isaac Newton studentship in astronomy and physical optics.

PROFESSOR L. CREMONA, of Rome, has been elected an associate of the Belgian academy of sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARISTOTELES. Aristoteles und die Arithmetik, von A. Görland. Marburg, 1898. 8vo. 61 pp. M. 1.80

BIANCHI (L.). Teoria delle funzioni di variabile complessa e delle funzioni ellittiche. Parte I.: Funzioni monodrome di variabile complessa. Pisa, 1899. 8vo. Fr. 8.50

BOBYNIN (V. V.). Bibliography of mathematical and physical sciences. Catalogue of books and memoirs of mathematical and physical sciences published in Russia from the invention of printing up to the present time. Part III (1800–1865). Moscow, 1898. 8vo. 171 pp. (Russian.)

BREEN (A. J. v.). Merkwaardige Punten en Lignen van den vlakken Driehoek. 2de vermeerde Druk. Amsterdam, 1898. 8vo. 88 pp. with 68 figures. M. 2.50

BROCARD (H.). Notes de bibliographie des courbes géométriques. Bar le-Duc, 1898. 22, 296 and 30 pp. (Autographié.) Fr. 3.00

COLLIN (K. R.). Lärobok i plan analytisk geometri. 2. upplaga. Stockholm, 1898. 8vo. 135 pp. M. 3.80

COLLINGWOOD (S. D.). See Dodgson (C. L.).

CLAUDEL (J.). Essai de synthétique. La méthode coordinative, la synthèse et l'enseignement intégral. Paris, 1899. 8vo Fr 4.50

DICKSTEIN (S.). See PASCAL (E.).

DODGSON (C. L.). The life and letters of Lewis Carroll (C. L. Dodgson), by S. D. Collingwood. London, 1898. 8vo. Cloth. 7s. 6d.

GERHARDT (E. J.). See LEIBNITZ (G. W.).

GÖRLAND (A.). See ARISTOTELES.

GUTZMER (A.). See JAHRESBERICHT.

HAUCK (G.). See JAHRESBERICHT.

HEIS (E.). Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra, in systematischer Folge bearbeitet. 98te Auflage. Köln, 1898. 8vo. 4 and 403 pp. M. 3.00

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. 2tes Heft: Die Kegelschnitte. Abteilung I. 2te Auflage. Leipzig, Teubner. 8vo.

JAHRESBERICHT der deutschen Mathematiker-Vereinigung, herausgegeben von G. Hauck und A. Gutzmer. Band VI, 2tes Heft. Leipzig, Teubner. 8vo. 4 and 110 pp. M. 4.00

——Dasselbe. Band VII, 1tes Heft. 159 pp. M. 4.80

KOCH (R.). Bestimmung der viergliedrigen Gruppen des Raumes (x, y, z). Leipzig, 1898. 8vo. 73 pp. M. 2.00

KOVALEVSKY (G.). Ueber eine Kategorie von Transformations-gruppen einer vierdimensionalen Mannigfaltigkeit. Leipzig, 1898. 8vo. 52 pp. M. 2.00

KUDRIUMOFF (W. J.). Elements of descriptive geometry. St Petersburg, 1898. 8vo. 404 pp. (Russian). M. 14.00

L. S. Considerazioni e teoremi sulla funzione proporzionalità, nel calcolo cosi elementare come differenzial ed integral. Milano, Ingegneri, 1898. 8vo. 47 pp.

LAGRANGE (J. L.). Zusätze zu Euler's Elementen der Algebra. Unbestimmte Analysis. Uebersetzt von A. J. v. Oettingen, herausgegeben von H. Weber. Leipzig, 1898. 8vo. 171 pp. Cloth. M. 2.60

LEIBNITZ (G. W.). Briefwechsel mit Mathematikern. Herausgegeben von E. J. Gerhardt. Band I. Berlin, 1898. 8vo. 28 and 761 pp. M. 28.00

LEMANN (A.). Beiträge zum mathematischen Unterricht in den Oberklassen. Mülhausen, 1898. 4to. 32 pp. M. 1.80

MARTIN (A.). About fifth-power numbers whose sum is a fifth power. (*Mathematical Magazine*, vol. II, no. 11.) Washington, 1898. 4to. Pp. 201–208.

MÜLLER (E.). Die Geometrie orientirter Kugeln nach Grassmann'schen Methoden. Königsberg, 1898. 8vo. 46 pp. M. 1.80

NITSCHE (G. A.). Ueber die Probleme der Biegung und der sphärischen Abbildung der Flächen. Leipzig, 1898. 8vo. 70 pp. M. 2.00

OETTINGEN (A. J. v.). See LAGRANGE (J. L.).

PASCAL (E.). Infinitesimal Calculus. Part III.: Calculus of variations. Translated into Polish by S. Dickstein. Warsaw, 1898. M. 6.00

SCHIFF (W.). Exercises and problems in differential and integral calculus. Part I. St. Petersburg, 1898. 8vo. 326 pp. (Russian.) M. 6.00

SCHMITZ (A.). Die rationalen ebenen Curven. Münnerstadt, 1898. 8vo. 51 pp. M. 2.00

SCHOUTEN (G. J.). Analytische Meetkunde in het platte Vlak. Delft, 1898. 8vo. 93 pp. M. 4.00

TAYLOR (F. G.). Introduction to differential and integral calculus and differential equations. London, Longmans, 1899. 8vo. 592 pp. 8s.

WAGNER (J.). Maasanalytische Studien. Habilitationsschrift. Leipzig, Leiner. 8vo. 123 pp. M. 1.50

WEBER (H.). See LAGRANGE (J. L.).

II. ELEMENTARY MATHEMATICS.

AGAPOFF (D. W.). New trigonometry. Second edition, Orenburg, 1898. 8vo. 144 pp. (Russian.) M. 3.00

AITOFF (D.). See ALEXANDROFF (I.).

ALEXANDROFF (I.). Problèmes de géométrie élémentaire, groupés d'après les méthodes à employer pour leur résolution. Traduit du russe, sur la 6e édition, par D. Aitoff. Paris, Hermann, 1899. 8vo. 11 and 155 pp.

AMES (A. F.). See MCLELLAN (J. A.).

BEARD (W. S.). See PENDLEBURY (C.).

BOWERS (T. F.). Monthly progress syllabus. Algebra, Stage 2: a practical periodical course arranged to meet new requirements. 8 periods. London, Simpkin, 1899. 8vo. 80 pp. 8d.

——The same, answers. 12mo. 22 pp. 8d.

BROWN (R.). Concise arithmetic for schools. Complete course required for University and other "Locals" and preliminary professional examinations. 3d edition, with answers. London, Simpkin, 1899. 12mo. 26 pp. 2s.

BURILLO de SANTIAGO (M.). Elementos de matematica. 2. edicion. Tomo II.: Geometria y trigonometria. Madrid, 1898. 4to. 344 pp.

ELEMENTOS de aritmética, con algunas nociones de algebra; por los Hermanos de las escuelas cristianes. Cuarta edicion. Tours, Mame, 1899. 16mo. 392 pp.

FRÈRES DES ÉCOLES CHRÉTIENNES. Exercices de calcul sur les quatre opérations fondamentales de l'arithmétique. Livre de l'élève. Paris, Poussielgue, 1899. 18mo. 8 and 72 pp.

HAGY (C. K.). Handy book of trigonometric tables; arranged by H. C. Whittaker. New York, J. J. McVey, 1899. 32mo. 2 and 57 pp. Cloth. $.20

KNIGHT (W. F.). A first algebra. London, Relfe, 1899. 12mo. 96 pp. 6d.

MCLELLAN (J. A.) and AMES (A. F.). The public school mental arithmetic, based on McLellan and Dewey's "Psychology of numbers." New York, The Macmillan Co., 1899. 16mo. 11 and 138 pp. Cloth. $.25

NONNIS-MARZANNO (F.). Le matematiche elementari esposte secondo il metodo storico. Trattato d'aritmetica ragionata esteso fino ai logaritmi e loro applicazioni, con più di 500 problemi. Seconda edizione, riveduta. Florence, 1898. 8vo. 7 and 660 pp. Fr. 6.50

PENDLEBURY (C.) and BEARD (W. S.). Shilling arithmetic. With answers. London, Bell, 1899. 12mo. 244 pp. 1s. 4d.

——The same. 12mo. 202 pp. 1s.

PINCHERLE (S.). Algebra elementare. Quarta edizione. Milano, 1899. 12mo. 8 and 210 pp. Fr. 1.75

——Geometria pura elementare. Quarta edizione. Milano, 1899. 12mo. 7 and 175 pp. Fr. 1.75

PRIZZO (G.). L'insegnamento della matematica nelle scuole primarie e popolari. Verona, 1898. 8vo. Fr. 3.00

SACHS (J.). Tafeln zum mathematischen Unterricht. Baden-Baden, 1898. 4to. 74 pp. M. 2.00

SCHOCHOR-TROTZKY. Arithmetical problems. Part I. Fourth edition. St. Petersburg, 1898. 8vo. 172 pp. (Russian.) M. 1.60

SLOTSCHANSKY (P.). Theorems and problems of elementary trigonometry. Odessa, 1898. 8vo. 119 pp. (Russian.) M. 3.50

WHITTAKER (H. C.) See HAGY (C. K.).

III. APPLIED MATHEMATICS.

BEACH (F. E.). See HASTINGS (C. S.).

BERRY (A.). A short history of astronomy. New York, Charles Scribner's Sons, 1899. 12mo. 31 and 440 pp. Cloth. $1.50

BINDER (W.). Die cotirte Darstellung auf einer Bildfläche nebst einem Vorschlage zur einheitlichen Bezeichnung in der darstellenden Geometrie. Thiel II. Wiener-Neustadt, 1898. 8vo. 16 pp., with table. M. 1.20

BOLTZMANN (L.). See MAXWELL (J. C.).

CAJORI (F.). A history of physics in its elementary branches, including the evolution of physical laboratories. New York, The Macmillan Company, 1899. 8vo. 8 and 322 pp. Cloth. $1.60

CLAUSIUS (R.). Ueber die bewegende Kraft der Wärme und die Gesetze, welche sich daraus für die Wärmelehre selbst ableiten lassen. Herausgegeben von M. Planck. Leipzig, 1898. 8vo. 55 pp. Cloth. M. 0.80

DELAUNAY (C.). Practische en theoretische Mechanica. Geheel op nieuw bewerkt door C. Krediet. Deel I. Leiden, 1898. 8vo. 8 and 329 pp. M. 5.50

DURAND (W. F.). The resistance propulsion of ships; treating of the various subdivisions of resistance, their computation, experimental determination, etc., the propulsive action of geometrical action, etc. New York, J. Wiley & Sons, 1898. 8vo. Cloth. $5.00

HASTINGS (C. S.) and BEACH (F. E.). A text-book of general physics, for use of colleges and scientific schools. Boston, Ginn & Co., 1899. 8vo. 5 and 768 pp. Leather. $2.95

HAYFORD (J. F.). Text-book of geodetic astronomy. New York, J. Wiley & Sons, 1898. 8vo. $3.00

HICKS (W. M.). Researches in vortex motion. Part III.: On spiral or gyrostatic vortex aggregates. London, 1898. 4to. 67 pp. 3s. 6d.

JAMIESON (A.). Text-book of applied mechanics. Arranged for science and art and other engineering students. Volume II. Second edition, enlarged. Diagrams, examination questions. London, Griffin, 1899. 8vo. 498 pp. 8s. 6d.

KLUMPNER (K.). Zeit- und Endgeschwindigkeitscurven. Leitmeritz, 1898. 8vo. 11 pp. M. 1.20

KREDIET (C.). See DELAUNAY (C.).

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MAXWELL (J. C.). Ueber physikalische Kraftlinien. Herausgegeben von L. Boltzmann. Leipzig, 1898. 8vo. 147 pp. Cloth. M. 2.40

MENGER (J.). Lehrbuch der darstellenden Geometrie. 2te Auflage. Vienna, 1899. 8vo. 4 and 234 pp. Bound. M. 3.00

MORGAN (C.). Elementary hydrostatics. London, Rivingtons, 1899. 12mo. 114 pp. 2s. 6d.

NADAL (J.). Théorie mathématique de la machine à vapeur ; action des parois. Paris, Dunod, 1898. 8vo. 114 pp.

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POINCARÉ (H.). Scientia. La théorie de Maxwell et les oscillations hertziennes. Chartres, Durand, 1899. 8vo. 80 pp.

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SABUDSKY (N.). Theory of probability and its application to shooting. St. Petersburg, 1898. 8vo. 445 pp. (Russian.)

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THOMSON (J. J.). See POYNTING (J. H.).

ZINGER (N.). Course in practical astronomy. St. Petersburg, 1898. 8vo. 289 pp. (Russian.)

——Course of higher geodesy. St. Petersburg, 1898. 8vo. 231 pp. (Russian.)